Dynamics of Cavity QED in Stochastic Field in interacting Fock space

P.K.Das
Physics and Applied Mathematics Unit
Indian Statistical Institute
203,B.T.Road,Calcutta-700035
e-mail:daspk@isical.ac.in

1 Interacting Fock space

As a vector space one mode interacting Fock space $\Gamma(\mathbb{C})$ [3,5] is defined by

$$\Gamma(\mathbb{C}) = \bigoplus_{n=0}^{\infty} \mathbb{C}|n\rangle \tag{1}$$

where $\mathbb{C}|n\rangle$ is called the n-particle subspace. The different n- particle subspaces are orthogonal, that is, the sum in (1) is orthogonal. The norm of the vector $|n\rangle$ is given by

$$\langle n|n\rangle = \lambda_n \tag{2}$$

where $\{\lambda_n\} > 0$. The norm introduced in (2) makes $\Gamma(\mathbb{C})$ a Hilbert space.

An arbitrary vector f in $\Gamma(\mathbb{C})$ is given by

$$f \equiv c_0|0\rangle + c_1|1\rangle + c_2|2\rangle + \ldots + c_n|n\rangle + \ldots \tag{3}$$

with $\|f\| = (\sum_{n=0}^{\infty} |c_n|^2 \lambda_n)^{1/2} < \infty$. we assume also that the sequence $\{\lambda_n\}$ satisfies the condition $\inf_{n\geq 0} \lambda_n^{1/n} > 0$.

We now define following actions on $\Gamma(\mathbb{C})$

$$\begin{array}{rcl} a^\dagger|n\rangle & = & |n+1\rangle \\ a|n+1\rangle & = & \frac{\lambda_{n+1}}{\lambda_n}|n\rangle \end{array} \tag{4}$$

$a^\dagger$ is called the *creation operator* and its adjoint a is called the *annihilation operator*. To define the annihilation operator we have taken the convention $0/0 = 0$.

The commutation relation of the operators then takes the form

$$[a, a^\dagger] = \frac{\lambda_{N+1}}{\lambda_N} - \frac{\lambda_N}{\lambda_{N-1}} \tag{5}$$

where N is the number operator defined by $N|n\rangle = n|n\rangle$.

1.1 Interaction of the Cavity and the External Field

We consider the interaction of an interacting single-mode of quantized field confined in a cavity with a noisy external field. Let $\mathcal{H}_{\mathcal{A}}$ and $\mathcal{H}_{\mathcal{B}}$ be Hilbert spaces of the cavity and the external field respectively. The composite system is expressed by the tensor product space $\mathcal{H}_{\mathcal{A}} \otimes \mathcal{H}_{\mathcal{B}}$. The total Hamiltonian is given by

$$H_{total} = H_A \otimes I_B + I_A \otimes H_B + H_{int} \tag{6}$$

where H_A describes the Hamiltonian of the cavity mode. This Hamiltonian may be further decomposed into two parts

$$H_A = H_{cav} + H. \tag{7}$$

Here H is the residual Hamiltonian determined by the optical medium in the cavity, referred to as a *free Hamiltonian.* H_B is the Hamiltonian of the external field.

The interaction Hamiltonian H_{int} consists of four terms. We drop the energy non conserving terms corresponding to the rotating wave approximation and obtain the simplified Hamiltonian as

$$H_{int}(t) = i\sqrt{\gamma}[a(t)b^+(t) - a^+(t)b(t)] \quad (8)$$

with

$$[b(t), b^+(t^{'})] = \delta(t - t^{'}) \quad (9)$$

and γ is a coupling constant. Here a is the annihilation operator of the cavity and b is the annihilation operator of the external field.

1.2 Quantum Stochastic Process

In order to describe quantum stochastic process we define first an operator

$$B_{in}(t, t_0) = \int_{t_0}^{t} b_{in}(s)ds \quad (10)$$

where $b_{in}(t)$ satisfies the commutation relation (9).This $b_{in}(t)$ represents the field immediately before it interacts with the system and we regard it as an *input* to the system.

Now,

$$\begin{aligned} & [B_{in}(t,t_0), B^+_{in}(t,t_0)] \\ = & [\int_{t_0}^{t} b_{in}(t^{'})dt^{'}, \int_{t_0}^{t} b^+_{in}(t^{''})dt^{''}] \\ = & \int_{t_0}^{t}\int_{t_0}^{t} [b_{in}(t^{'}), b^+_{in}(t^{''})]dt^{'}dt^{''} \\ = & \int_{t_0}^{t}\int_{t_0}^{t} \delta(t^{'} - t^{''})dt^{'}dt^{''} \\ = & \int_{t_0}^{t}(\int_{t_0}^{t} \delta(t^{'} - t^{''})dt^{''})dt^{'} \\ = & \int_{t_0}^{t} dt^{'} \\ = & t - t_0 \end{aligned} \quad (11)$$

Now, we write down the increments

$$\begin{aligned} dB_{in}(t) &= B_{in}(t + dt) - B_{in}(t), \\ dB^+_{in}(t) &= B^+_{in}(t + dt) - B^+_{in}(t) \end{aligned} \quad (12)$$

Then

$$B_{in}(t + dt, t) = B_{in}(t + dt) - B_{in}(t) = dB_{in}(t) \quad (13)$$

and

$$B^+_{in}(t + dt, t) = B^+_{in}(t + dt) - B^+_{in}(t) = dB^+_{in}(t) \quad (14)$$

From (11), (12), (13) and (14) we get

$$[dB_{in}(t), dB^+_{in}(t)] = dt. \quad (15)$$

This leads to the natural definition of quantum stochastic process as

$$\begin{aligned} dB_{in}(t)dB^+_{in}(t) &= (N^{'} + 1)dt \\ dB^+_{in}(t)dB_{in}(t) &= N^{'}dt \\ dB_{in}(t)dB_{in}(t) &= Mdt \\ dB^+_{in}(t)dB^+_{in}(t) &= M^*dt \end{aligned} \quad (16)$$

and all other products higher than the second order in dB_{in} are equal to zero. $N^{'}$ and M are real and complex numbers satisfying

$$N^{'}(N^{'} + 1) \geq |M|^2. \quad (17)$$

2 Dynamics of Cavity QED in Stochastic Field

The evolution of an arbitrary operator X is given by

$$X(t) = U^+(t)XU(t) \quad (18)$$

in which the unitary operator $U(t)$ is generated by the Hamiltonian in (6). The Hamiltonians H_{cav} and H_B drive the cavity and the external field respectively. We shall assume here H to be zero. The unitary operator of the system is then given by

$$U(dt) = e^{\sqrt{\gamma}(adB^+_{in} - a^+dB_{in})} \quad (19)$$

The increment of an arbitrary operator r of the system driven by the stochastic input b_{in} is given by

$$\begin{aligned} & dr(t) \\ = & \sqrt{\gamma}[a^+dB_{in} - adB^+_{in}, r(t)]+ \\ & +\tfrac{\gamma}{2}\{(N^{'} + 1)(2a^+ra - a^+ar - ra^+a) \\ & +N^{'}(2ara^+ - aa^+r - raa^+) \\ & +M[a^+, [a^+, r]] + M^*[a, [a, r]]\}dt \end{aligned} \quad (20)$$

The dynamical behaviour of the optical cavity in interacting Fock space is now described on replacing the general operator $r(t)$ in equation (20) by the operator $a(t)$ of the QED bath. Then using the commutation relation (15) and the stochastic process given in [1] we get

$$\begin{aligned} da &= a(t+dt) - a(t) \\ &= \{-\tfrac{\gamma}{2}(\tfrac{\lambda_{N+1}}{\lambda_N} - \tfrac{\lambda_N}{\lambda_{N-1}})a \\ &\quad -\sqrt{\gamma}(\tfrac{\lambda_{N+1}}{\lambda_N} - \tfrac{\lambda_N}{\lambda_{N-1}})b_{in}(t)\}dt \end{aligned} \tag{21}$$

This implies

$$\begin{aligned} \dot{a}(t) &= -\tfrac{\gamma}{2}(\tfrac{\lambda_{N+1}}{\lambda_N} - \tfrac{\lambda_N}{\lambda_{N-1}})a(t) \\ &\quad -\sqrt{\gamma}(\tfrac{\lambda_{N+1}}{\lambda_N} - \tfrac{\lambda_N}{\lambda_{N-1}})b_{in}(t) \end{aligned} \tag{22}$$

The state equation represented by (22) of the dynamics of the cavity A in the interacting Fock space is a generalization of the well known Langevin equation of the cavity in boson Fock space [1, 2, 8]. In case of boson Fock space the commutator described in equation (5) becomes unity, that is,

$$[a, a^*] = \frac{\lambda_{N+1}}{\lambda_N} - \frac{\lambda_N}{\lambda_{N-1}} \equiv \Lambda_N^{'} = 1$$

The dynamics of the cavity given by (22) then reduces to the usual quantum Langevin form in boson Fock space

$$\dot{a}(t) = -\frac{\gamma}{2}a(t) - \sqrt{\gamma}b_{in}(t). \tag{23}$$

Due to interaction of the evolving incoming field with the optical cavity an outgoing field is produced and is given by

$$B_{out}(t, t_0) = \int_{t_0}^{t} b_{out}(s)ds, \tag{24}$$

$$b_{out}(t) = U^{+}(dt)b_{in}(t)U(dt) \tag{25}$$

The input-output relation after the interaction at time t is given simply by the following derivation:

$$\begin{aligned} dB_{out}(t) &= U^{+}(dt)dB_{in}(t)U(dt) \\ &= dB_{in}(t) + \sqrt{\gamma}a[dB_{in}, dB_{in}^{+}] \end{aligned} \tag{26}$$

Now using (15) the above relation gives us

$$b_{out}(t)dt = b_{in}(t)dt + \sqrt{\gamma}adt \tag{27}$$

and hence we have the required input-output relation of the cavity QED

$$b_{out}(t) = \sqrt{\gamma}a(t) + b_{in}(t). \tag{28}$$

We have seen that the cavity dynamics may be thought of as an operator equation in Hilbert space. The equations (22) and (28) give the state equation and the system output of a cavity in different modes. The state equation of the cavity dynamics along with the output equation can be represented with usual notations in the general form of classical control system as

$$\begin{aligned} \dot{a}(t) &= A^{'}a(t) + B^{'}b_{in}(t) \\ b_{out}(t) &= C^{'}a(t) + D^{'}b_{in}(t) \end{aligned} \tag{29}$$

where

$$\begin{aligned} A^{'} &= -\tfrac{\gamma}{2}(\tfrac{\lambda_{N+1}}{\lambda_N} - \tfrac{\lambda_N}{\lambda_{N-1}}) \\ B^{'} &= -\sqrt{\gamma}(\tfrac{\lambda_{N+1}}{\lambda_N} - \tfrac{\lambda_N}{\lambda_{N-1}}) \\ C^{'} &= \sqrt{\gamma} \\ D^{'} &= 1 \end{aligned} \tag{30}$$

The dynamics of the cavity in the interacting Fock space is a first order differential equation of the system operators with variable space parameter $\Lambda_N^{'}$ as coefficient.

Applying Laplace transform in (29), assuming zero initial state of the QED bath, we get the transfer function representation of the optical QED system in interacting mode as,

$$b_{out}(s) = G(s)b_{in}(s), \tag{31}$$

$$G(s) = \frac{s - \frac{\gamma_A}{2}\Lambda_N^{'}}{s + \frac{\gamma_A}{2}\Lambda_N^{'}}. \tag{32}$$

We have seen that a cavity QED in interacting mode in some way closely analogous to the classical one in which the input and the output are described by operators in Hilbert space.

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