Multiple–Attractor Bifurcations and Quasiperiodicity in Nonsmooth Systems

Zhanybai T. Zhusubaliyev, Erik Mosekilde and Soumitro Banerjee

Piecewise-smooth maps typically arise as discrete-time models of dynamical systems when the continuous evolution in time is punctuated by impacts or discrete switching events that alter the form of the constitutive equations. Examples of such systems include power electronic converters and switching circuits [1], [2], [3], mechanical systems with impacts and friction [4] as well as models of certain physiological [5] and economic systems [6]. As a parameter is varied, the fixed point for the Poincaré map of such a system may move in phase space and collide with the border between two smooth regions. When this happens, the eigenvalues of Jacobian matrix can change abruptly, leading to a special class of nonlinear dynamic phenomena known as border-collision bifurcations [7], [8], [9].

In a couple of recent publications, Kapitaniak and Maistrenko [10] and Duta *et al.* [11] showed that piecewise-smooth systems can exhibit a special type of border-collision bifurcation in which several attractors are created simultaneously. A main feature of this type of bifurcation is that, close to the bifurcation point, the distance between the basins of attraction may be arbitrarily small. In the presence of noise, no matter how small, this leads to a fundamentally unpredictable behavior of the system when a system parameter is slowly varied through the bifurcation point.

Many physical systems, including switching circuits and impact oscillators, are known to display quasiperiodicity and other forms of multimode dynamics [4], [3], [12], [13].

In our recent work [14], [15], [16] we have shown that border-collision bifurcations can lead to the birth of a stable closed invariant curve associated with a quasiperiodic or a periodic orbit. This phenomenon resembles the well-known Neimark-Sacker bifurcation in several respects. However, rather than through a continuous crossing of a pair of complex-conjugate multipliers of the periodic orbit through the unit circle, the border-collision bifurcation involves a jump of the multipliers from the inside to the outside of this circle.

This leads to the questions: Can piecewise-smooth systems exhibit border-collision bifurcations in which a stable periodic orbit arises together with an attracting closed invariant curve? And, if this is the case what is the mechanism for the particular type of multiple-attractor bifurcation?

In order to address these questions, we first consider the normal form map that represents the behavior of the piecewise-smooth systems in a close neighborhood of the border.

$$f : \begin{pmatrix} x \\ y \end{pmatrix} \mapsto \begin{cases} f^-(x,y), & x \le 0; \\ f^+(x,y), & x > 0, \end{cases} \tag{1}$$

where

$$f^-(x,y) = \begin{pmatrix} \tau^- x + y + \mu \\ -\delta^- x \end{pmatrix};$$

$$f^+(x,y) = \begin{pmatrix} \tau^+ x + y + \mu \\ -\delta^+ x \end{pmatrix}, \quad (x,y) \in \mathbb{R}^2.$$

In this representation the phase plane is divided into two regions, $D^- = \{(x,y) : x \le 0, y \in \mathbb{R}\}$ and $D^+ = \{(x,y) : x > 0, y \in \mathbb{R}\}$. τ^- and δ^- denote, respectively, the trace and the determinant of the Jacobian matrix in the half-plane D^-, and τ^+ and δ^+ are the trace and determinant of the Jacobian matrix in D^+.

The theory of border-collision bifurcations developed so far assumes contractive dynamics on both sides of the discontinuity (i.e., $|\delta^-| < 1$ and $|\delta^+| < 1$)) [17], [18]. We consider a situation where an attracting fixed point changes into a spirally repelling fixed point as it moves across the border. This is ensured by assuming $\delta^- < 1$, $\delta^+ > 1$, with

$$-(1+\delta^-) < \tau^- < 1+\delta^- \text{ and } -2\sqrt{\delta^+} < \tau^+ < 2\sqrt{\delta^+} \tag{2}$$

If $\mu < 0$ then the map (1) has a single nontrivial stable fixed point with a negative x-coordinate. When μ changes sign, the x-coordinate of the fixed point also changes sign and the fixed point abruptly loses stability when a pair of complex-conjugate eigenvalues of the Jacobian matrix jump from the inside to the outside of the unit circle, i.e. the stable focus transforms into an unstable focus. This transition leads to the birth of a stable invariant curve, associated with quasiperiodic or phase-locked dynamics.

However, more complicated bifurcation phenomena are also possible in such transitions. To study these phenomena, we have calculated the chart of dynamical modes in the parameter plane (τ^-, τ^+) for positive values of $\mu > 0$ (Fig. 1). As shown in Fig. 1, this chart is characterized by a dense set

The work was supported by the Russian Foundation for Basic Research (grant 06-01-00811a) and by the Danish Natural Science Foundation through the Center for Modelling, Nonlinear Dynamics, and Irreversible Thermodynamics (MIDIT).

Zh. T. Zhusubaliyev is with the Department of Computer Science, Kursk State Technical University, 50 Years of October Str., 94, Kursk 305040, Russia zhanybai@mail.kursk.ru.

E.Mosekilde is with the Complex Systems Group, Department of Physics, The Technical University of Denmark, 2800 Lyngby, Denmark Erik.Mosekilde@fysik.dtu.dk.

S. Banerjee is with the Department of Electrical Engineering and the Centre for Theoretical Studies, Indian Institute of Technology, Kharagpur-721302, India soumitro.banerjee@gmail.com.

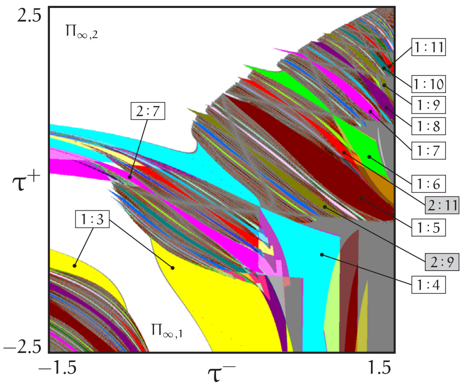

Fig. 1. Chart of dynamical modes in the parameter plane (τ^-, τ^+) with the remaining parameters fixed at $\delta^- = 0.5$, $\delta^+ = 1.6$, and $\mu = 0.05$.

of periodic tongues. Main resonance tongues in the figure are marked with the corresponding rotation numbers. Between the tongues there are parameter combinations that lead to quasiperiodicity. In the chart of dynamical modes shown in Fig. 1, $\Pi_{\infty,1}$ and $\Pi_{\infty,2}$ are domains where the trajectories of the map diverge to infinity for all initial conditions.

Within each tongue of periodicity there is a closed invariant curve that is formed by the unstable manifold of the saddle cycle and the points of the stable (node or focus) and saddle cycles. With changes of the parameters τ^- and τ^+, this invariant curve is destroyed through a homoclinic bifurcation. However, the stable and saddle cycles continue to exist after the destruction of the torus. With further change of the parameters, these cycles merge and disappear in a border-collision fold bifurcation. As a result, between the curves of homoclinic bifurcation and of border-collision fold bifurcation, periodic and quasiperiodic attractors coexist. On the boundaries of this region one can observe hysteretic transitions.

For parameter values corresponding to operation conditions between these curves, as the parameter μ crosses the bifurcation point at $\mu = 0$, a multiple-attractor bifurcation takes place in which a quasiperiodic orbit arises together with stable and saddle fixed points. This is typical of regions in the parameter plane where tongues of different dynamical modes overlap.

After this stable closed invariant curve has been destroyed, the multiple attractor bifurcation within the regions of intersection results in the simultaneous appearance of several pairs of stable and saddle cycles, none of which are situated on an invariant curve. In other cases, the destruction of the stable closed invariant curve leads to the appearance of a chaotic attractor that can coexist with a stable cycle. For particular parameter values one can then observe a multiple-attractor bifurcation in which chaotic and periodic attractors are simultaneously created from the stable focus fixed point in a border-collision bifurcation.

Considering a two-dimensional piecewise-smooth map describing the behavior of a DC/DC converter with two-level pulse-width modulated control, we show that the phenomena observed for the piecewise linear normal form map may actually occur in practical systems.

We obtain the chart of dynamical modes in the relevant parameter plane through a detailed analytical and numerical study and demonstrate how torus birth can take place either via a classical Neimark–Sacker bifurcation or via a border-collision bifurcation. Next, we discuss how the torus is destroyed through a homoclinic bifurcation. This analysis involves the use of numerical methods that can follow the stable and unstable manifolds for the various modes.

Finally, we illustrate how the normal form theory can predict the bifurcation behavior along the the border-collision torus birth bifurcation curve. We obtain the functional relationships between the parameters of the normal form map (1) and actual system parameters analytically and study the local character of the bifurcation behavior of this system.

References

[1] S. Banerjee and G. C. Verghese, Eds., *Nonlinear Phenomena in Power Electronics: Attractors, Bifurcations, Chaos, and Nonlinear Control*. IEEE Press, 2001.

[2] C. K. Tse, *Complex Behavior of Switching Power Converters*. Boca Raton, USA: CRC Press, 2003.

[3] Z. T. Zhusubaliyev and E. Mosekilde, *Bifurcations and Chaos in Piecewise-Smooth Dynamical Systems*. Singapore: World Scientific, 2003.

[4] R. I. Leine and H. Nijmeijer, *Dynamics and Bifurcations of Non-Smooth Mechanical Systems*. Berlin: Springer Verlag, 2004.

[5] J. Keener and J. Sneyd, *Mathematical Physiology*. New York: Springer Verlag, 1998.

[6] J. Laugesen and E. Mosekilde, "Border-collision bifurcations in a dynamic management game," *Comp. Oper. Res.*, vol. 33, pp. 464–478, 2006.

[7] M. I. Feigin, *Forced Oscillations in Systems with Discontinuous Nonlinearities*. Moscow: Nauka Publ., 1994, in Russian.

[8] H. E. Nusse and J. A. Yorke, "Border-collision bifurcations including "period two to period three" for piecewise smooth maps," *Physica D*, vol. 57, pp. 39–57, 1992.

[9] M. di Bernardo, M. I. Feigin, S. J. Hogan, and M. E. Homer, "Local analysis of C-bifurcations in n-dimensional piecewise smooth dynamical systems," *Chaos, Solitons & Fractals*, vol. 10, no. 11, pp. 1881–1908, 1999.

[10] T. Kapitaniak and Yu. L. Maistrenko, "Multiple choice bifurcations as a source of unpredictability in dynamical systems," *Phys. Rev. E*, vol. 58, no. 4, pp. 5161–5163, 1998.

[11] M. Dutta, H. E. Nusse, E. Ott, J. A. Yorke, and G.-H. Yuan, "Multiple attractor bifurcations: A source of unpredictability in piecewise smooth systems," *Phys. Rev. Lett.*, vol. 83, pp. 4281–4284, 1999.

[12] A. El Aroudi and L. Benadero and E. Toribio and S. Machiche, "Quasiperiodicity and chaos in the dc-dc buck-boost converter," *Int. J. Bif. Chaos*, vol. 10, no. 2, pp. 359–371, 2000.

[13] H. Dankowicz, P. Piiroinen, and A. B.Nordmark, "Low-velocity impacts of quasiperiodic oscillations," *Chaos, Solitons & Fractals*, vol. 14, no. 2, pp. 241–255, 2002.

[14] Zh. T. Zhusubaliyev and E. Mosekilde, "Birth of bilayred torus and torus breakdown in a piecewise-smooth dynamical system," *Phys. Lett. A*, vol. 351, no. 3, pp. 167–174, 2006.

[15] ——, "Torus birth bifurcation in a DC/DC converter," *IEEE Trans. Circ. Syst. I*, vol. 53, no. 8, pp. 1839–1850, 2006.

[16] Zh. T. Zhusubaliyev, E. Mosekilde, S. M. Maity, S. Mohanan, and S. Banerjee, "Border collision route to quasiperiodicity: Numerical investigation and experimental confirmation," *Chaos*, vol. 16, p. 023122, 2006.

[17] S. Banerjee and C. Grebogi, "Border collision bifurcations in two-dimensional piecewise smooth maps," *Phys. Rev. E*, vol. 59, no. 4, pp. 4052–4061, 1999.

[18] S. Banerjee, P. Ranjan, and C. Grebogi, "Bifurcations in two-dimensional piecewise smooth maps — theory and applications in switching circuits," *IEEE Trans. Circ. Syst. I*, vol. 47, no. 5, pp. 633–643, 2000.