On game-theoretical modeling with applications to the problems of large-scale competitive projects

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Abstract—**The paper is devoted to game-theoretical control problems motivated by economic decision making situations arising in realization of large-scale projects, such as designing and putting into operations the new gas or oil pipelines. The approaches and models discussed in the paper seem to be also useful in the fields of physics, biology and other natural sciences.**

I. Introduction

The problem description is the following. Let us assume that there is a market with increasing demand for some goods (say, natural gas or oil). One can supply goods produced (delivered) as a result of realization of a project (for example, to supply gas to this market by constructing the corresponding gas pipelines). Evidently, appearance of the new participants at the market, while saturating it, leads to a decrease in sales for the existing participants. It means that the earlier a participant enters the market the greater profit he or she gets. At the same time, the present value of the cost of project realization is decreasing, and also the demand and prices may be increasing, therefore the later entering the market might be preferable. The above arguments lead to a game-theoretical problem, in which the moments of time when each participant enters the market, play crucial role.

The detailed investigation of the problem for the case of gas pipeline construction included mathematical and computer modeling of Turkey's gas market development. In the paper [1] a rigorous mathematical model was proposed, where the above problem was formalized as a non-cooperative game in which the moments of entering the market (commercialization times) were taken as control variables. Integral payoff functional for each participant (Player) was defined as a future profit determined in a corresponding way for the whole operating period. A software based on this model presented in [2].

In the following research attempts have been made to extend the developed approach and obtained results to modeling the China's natural gas market. However, most of the assumptions admissible for Turkey's gas market turned out to be unfit for specific China's market. The main economic distinction is that price formation mechanism can not be considered as purely market in this case. Problem description and numerical results for real data related to the planned gas pipelines from Russia are presented in the paper [3].

In the present paper we propose a mathematical model that takes into account the features of not purely market economy. That leads to the fact that some assumptions of the model are significantly different and sometimes are opposed to those accepted in the in [1]. The main features of the model are the following. The problem is considered within finite horizon, profit rate return function are supposed to be monotonously increasing rather than decreasing, cost of construction is constant for each participant. Criteria functions, which are to be minimized are defined as combinations of return of investment time and time of entering the market.

II. Problem Formulation

The precise problem formulation is the following. For the case of two participants the benefit rate function for each of them is given:

$$\varphi_1(t\,|\,t_2)=\begin{cases}\varphi_{11}(t), & if \quad t<t_2\\ \varphi_{12}(t), & if \quad t\ge t_2\end{cases},$$

$$\varphi_2(t\,|\,t_1)=\begin{cases}\varphi_{21}(t), & if \quad t<t_1\\ \varphi_{22}(t), & if \quad t\ge t_1\end{cases}.$$

The functions defined above determine the benefit of the participants for each fixed time interval. Thus, the profit of the first participant during period $[t,t+\delta]$ is $\int_{t_1}^{t_1+\delta}\varphi_1(t\,|\,t_2)dt$ and depends on the moment when the second player enters the market. Analogously the profit of the second participant is determined. We assume the functions $\varphi_{ij}(t)$ to be continuous, concave and monotonously increasing on [0,T]. Besides $\varphi_{i1}(t)>\varphi_{i2}(t),\ i=1,2;\quad t\in[0,T]$.

We also assume that realization costs of the projects (construction costs of pipeline) are fixed and denoted by C_i , $i=1,2$. The value $\Delta_i=\Delta_i(t_1,t_2)$ that is defined by the equality

$$\int_{t_i}^{t_i+\Delta_i}\varphi_i(t\,|\,t_j)dt=C_i \qquad (1)$$

is called the payback period of the project i . Here $j=1,2$; $j\neq i$.

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Problem statement considered in this paper supposes that each participant tries to achieve two goals: to minimize their payback period $\Delta_i = \Delta_i(t_1, t_2)$ and to minimize the commercialization time of the project, that is the time t_i of entering the market. Preferences of participants related to priorities between these two criteria may be different and are determined by the choice of weights. Control variables for each participant are (as in cited papers [1-3]) the commercialization times t_i .

According to slightly simplified formulation of this problem we define payoff functions that are to be minimized by the choice of commercialization times t_i as follows:

$$f_i(t_1, t_2 \mid \alpha_i) = \alpha_i t_i + \Delta_i(t_1, t_2) ,$$

where α_i is a weight coefficient, $0 \le \alpha_i \le 1$. With $\alpha_i = 1$ both criteria are equitable, in case $\alpha_i = 0$ one has unique criterion - the payback period.

Thus, in the paper we consider the following problems.

Problem 1. Construct and investigate the function $t_1^0 = t_1^0(t_2 \mid \alpha_1)$ (in general - multivalued) defined by the equation

$$f_1(t_1^0, t_2 \mid \alpha_1) = \min_{t_1} f_1(t_1, t_2 \mid \alpha_1) . \qquad (2)$$

According to standard terminology the function $t_1^0 = t_1^0(t_2 \mid \alpha_1)$ is called the function of best replies of the participant 1 to the choice t_2 of the second participant. Similarly, the function $t_2^0 = t_2^0(t_1 \mid \alpha_2)$ is defined - the function of best replies of the participant 2 to the choice t_1 of the first participant.

The second problem under consideration relates to finding Nash equilibrium solutions in corresponding two player game. Using the introduced terminology it can be formulated in the following way.

Problem 2. Find pairs $\{\bar{t}_1, \bar{t}_2\}$ that satisfy the conditions

$$\bar{t}_1 \in t_1^0(\bar{t}_2 \mid \alpha_1) ,$$
$$\bar{t}_2 \in t_2^0(\bar{t}_1 \mid \alpha_2) .$$

The properties of the above problems and results of computer simulations are presented in the paper.

III. Properties of the Project's Payback Periods $\Delta_i = \Delta_i(t_1, t_2)$

Let us introduce the following notations that correspond to some specific moments of time.

Define $\bar{t}_i, \bar{\bar{t}}_i, t_i^*, t_i^{**}$ by the following relations.

$$\bar{t}_i : \int_0^{\bar{t}_i} \varphi_{i1}(t)dt = C_i; \quad \bar{\bar{t}}_i : \int_{\bar{\bar{t}}_i}^{T} \varphi_{i2}(t)dt = C_i;$$

$$t_i^* : \int_0^{t_i^*} \varphi_{i2}(t)dt = C_i; \quad t_i^{**} : \int_{t_i^{**}}^{T} \varphi_{i1}(t)dt = C_i .$$

In the following we assume that the final time T is large enough in comparison with time characteristics of the projects. Namely, the following assumption is supposed to be true.

Assumption 1. The inequalities hold true:

$$t_i^* < \bar{\bar{t}}_i , \quad i = 1, 2.$$

Remark that under above conditions the following inequalities are true: $0 < \bar{t}_i < t_i^* < \bar{\bar{t}}_i < t_i^{**} < T$.

In the following we will consider the function $\Delta_1 = \Delta_1(t_1, t_2)$. Similar results for the function $\Delta_2 = \Delta_2(t_1, t_2)$ can be obtained by appropriate changing of indices.

Introduce the following functions of the argument $\tau \in [0, T]$. Assign

$$g_1 = g_1(\tau): \quad \int_{\tau}^{\tau + g_1} \varphi_{11}(t)dt = C_1;$$

$$g_2 = g_2(\tau): \quad \int_{\tau}^{\tau + g_2} \varphi_{12}(t)dt = C_1.$$

Assumption 2. For each possible t_1 the following inequality is true: $\varphi_{11}(t_1) < \varphi_{12}(t_1 + g_1(t_1))$.

This condition restricts the relationship between benefit rates $\varphi_{11}(t_1)$, $\varphi_{12}(t_1)$ and cost of construction C_1. It is true when payback period of the project is sufficiently large in comparison with changing of the functions $\varphi_{1j}(t)$ near the point t_1.

To formulate the properties of the function $\Delta_1 = \Delta_1(t_1, t_2)$ it is useful to introduce one more point $t_1^{'} = t_1^{'}(t_2)$. This point for a fixed value t_2 is defined by relation

$$\int_{t_1^{'}}^{t_2} \varphi_{11}(t)dt = C_1.$$

Remark that the point $t_1^{'} = t_1^{'}(t_2)$ is defined and non-negative for $t_2 \ge \bar{t}_1$.

The domain $D_1 \subset [0,T] \times [0,T]$ where the function $\Delta_1 = \Delta_1(t_1, t_2)$ is correctly defined is determined by the following relations: $0 \le t_1 \le z(t_2)$, $0 \le t_2 \le T$. Here $z_2 = z(t_2)$ is such a point that $\int_{z_2}^{T} \varphi_1(t \mid t_2)dt = C_1.$

Under assumptions 1-2 the function $\Delta_1 = \Delta_1(t_1, t_2)$ has the following properties.

Lemma 1: Within the set D_1 the function $\Delta_1 = \Delta_1(t_1, t_2)$ is defined correctly and is continuous. In the interior points of this set with $t_1 \neq t_1^{'}(t_2)$ and $t_1 \neq t_2$ there exists partial derivative with respect to variable t_1 that satisfies the relation

$$\frac{\partial \Delta_1(t_1,t_2)}{\partial t_1} = \begin{cases} -1 + \dfrac{\varphi_{11}(t_1)}{\varphi_{11}(t_1 + g_1(t_1))}, & if \;\; 0 < t_1 < t_1^{'}(t_2) \\ -1 + \dfrac{\varphi_{11}(t_1)}{\varphi_{12}(t_1 + \Delta_1(t_1,t_2))}, & if \;\; t_1^{'}(t_2) < t_1 < t_2 \\ -1 + \dfrac{\varphi_{12}(t_1)}{\varphi_{12}(t_1 + g_2(t_1))}, & if \;\; t_2 < t_1 < z(t_2) \end{cases}$$

Lemma 2: When assumptions 1-2 hold true the function $\Delta_1 = \Delta_1(t_1, t_2)$ as a function of variable t_1 decreases in the interval $[0, z(t_2)]$. The derivative $\dfrac{\partial \Delta_1(t_1,t_2)}{\partial t_1}$<0 and it is continuous within this interval excluding two points $t_1 = t_1^{'}(t_2)$ and $t_1 = t_2$. At the first point the derivative increases and at the second - decreases. For each $(t_1, t_2) \in \operatorname{int} D_1$ the inequalities are true:

$$g_1(t_1) \leq \Delta_1(t_1,t_2) \leq g_2(t_1)\,.$$

Graphical illustration of the above assertions for the example that will be considered in the last section of the paper are presented in Fig. 1. In Fig.1a there is a graphic of benefit rate $\varphi_1(t \mid t_2)$ with $t_2 \in (\bar{t}_1, \bar{\bar{t}}_2)$ being fixed. The area of the shaded figure is C_1 and its base is $\Delta_1(t_1,t_2)$ = 10.76 for arguments t_1=15, t_2=20. In Fig.1b the graph of the function $\Delta_1 = \Delta_1(t_1,t_2)$ for t_2=20 is presented.

a b

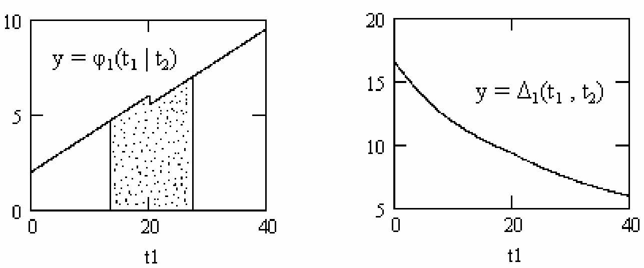

Fig. 1. Benefit rate $\varphi_1(t \mid t_2)$ for t_2=20$\in (\bar{t}_1, \bar{\bar{t}}_2)$ and corresponding $\Delta_1(t_1,t_2)$ = 10.76

a b

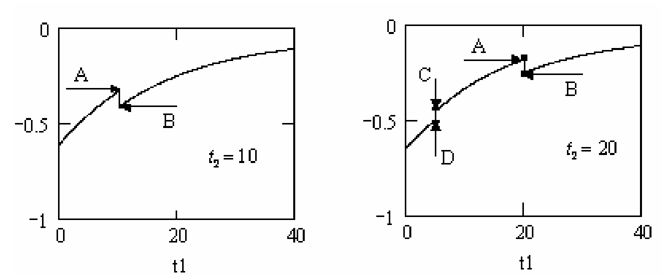

Fig.2. Derivative of the function $\Delta_1 = \Delta_1(t_1,t_2)$ for two different values of t_2.

Fig. 2 illustrates the assertion of *Lemma 2.* Here the points A – D correspond to the points of discontinuity. For the example considered in the section VI there may be at most two such points.

Closing this section we propose an assertion that characterizes the value $\Delta_1 = \Delta_1(t_1,t_2)$ as a function of argument t_2.

Lemma 3: On the assumptions 1-2 the function $\Delta_1 = \Delta_1(t_1,t_2)$ as a function of argument t_2 has the following properties. Let the point $t_1 \in [0, \bar{\bar{t}}_1]$ be fixed. Then in the interval $[0, t_1]$ the function is constant: $\Delta_1(t_1,t_2) \equiv g_2(t_1)$, in the interval $[t_1, t_1 + g_1(t_1)]$ it decreases to the value $g_1(t_1)$, then, with $t_2 > t_1 + g_1(t_1)$ it is constant again: $\Delta_1(t_1,t_2) \equiv g_1(t_1)$. If $t_1 \in [\bar{\bar{t}}_1, t_1^{**}]$, then $\Delta_1(t_1,t_2)$ is defined only for $t_2 : \int_{t_2}^{T} \varphi_1(t \mid t_2) dt \geq C_1$, decreases when $t_2 < t_2^{**}$ and is constant on $[t_1^{**}, T]$.

IV. Best Reply Functions

As in the previous section we consider the problem from the point of view of the first participant. The goal of this section is to construct the function of best replies $t_1^0 = t_1^0(t_2 \mid \alpha_1)$ of the first participant for the choice t_2 of the second one, i.e. a solution of the *Problem* 1. The values of this function are defined by equation (2).

Let us introduce the following notations:

$$p_{11}(t) = \frac{\varphi_{11}(t)}{\varphi_{11}(t + g_1(t))}, \qquad p_{12}(t) = \frac{\varphi_{12}(t)}{\varphi_{12}(t + g_2(t))},$$

$$q_{11}(t) = \frac{\varphi_{11}(t)}{\varphi_{12}(t + g_1(t))}, \qquad q_{12}(t) = \frac{\varphi_{11}(t)}{\varphi_{12}(t + g_2(t))}\,.$$

Lemma 4: Under conditions of assumptions 1-2 the functions $p_{11}(t), p_{12}(t), q_{11}(t), q_{12}(t)$ are continuous and monotonously increasing in their definition domains. The inequalities are true:

$$q_{11}(t) > q_{12}(t) > p_{12}(t), \quad q_{11}(t) > p_{11}(t) \qquad (3)$$

for each possible t.

Let us fix a value α_1 such that

$$q_{11}(0) < 1-\alpha_1 < p_{1j}(t_1^*) \quad (j=1,2),$$

and define the points t_1^-, t_1^+, t_1^q and t_2^q as solutions to equations

$$t_1^-: \quad p_{11}(t) = 1-\alpha_1, \quad t_1^+: \quad p_{12}(t) = 1-\alpha_1,$$

$$t_1^q: \quad q_{11}(t) = 1-\alpha_1, \quad t_2^q: \quad q_{12}(t) = 1-\alpha_1.$$

Remark that such $0 < \alpha_1 < 1$ do exist and the roots of above equations are defined unambiguously.

For defined $0 < \alpha_1 < 1$ we can construct the function $t_1^0 = t_1^0(t_2 \mid \alpha_1)$ (multi-valued at some specific points). Due to inequalities (3) we have $t_1^q < t_2^q < t_1^+$ and $t_1^q < t_1^-$. For a position of t_1^- with respect to t_1^+ and t_2^q there are several opportunities. In the following we will deal with the case:

$$t_1^q < t_2^q < t_1^- < t_1^+ \qquad (4)$$

Other dispositions of t_1^- can be studied similarly.

Theorem 1: The following relations are true:

a) If $0 < t_2 < t_2^q$, then $t_1^0(t_2 \mid \alpha_1) = t_1^+$.

b) When $t_2 \geq t_2^q$, but $t^{'}(t_2) < t_1^q$ the set $t_1^0(t_2 \mid \alpha_1)$ consists of one or two points: $\gamma_1(t_2)$ and t_1^+, where $\gamma_1(t_2)$ is the unique solution to equation

$$\frac{\varphi_{11}(\gamma_1)}{\varphi_{11}(\gamma_1 + \Delta_1(\gamma_1, t_2))} = 1-\alpha_1 \qquad (5)$$

that moves from the point t_2^q to the point t_1^q.

c) When $t^{'}(t_2) = t_1^q$ one has $\Delta_1(t_1, t_2) = g_1(t_1)$, and the root of the equation (5) coincides with the point t_1^q. From this point, on the interval $t_2^q < t^{'}(t_2) < t_1^-$ the set $t_1^0(t_2 \mid \alpha_1)$ consists of the unique point $t_1^{'}(t_2)$.

d) When t_2 is such that $t^{'}(t_2) > t_1^-$ the set $t_1^0(t_2 \mid \alpha_1)$ consists of the unique point t_1^-.

V. Equilibrium Solutions

Because of the fact that there is an interval in the definition domain, in which the function $t_i(t_j) = t_i^0(t_j \mid \alpha_i), (i, j = 1, 2; i \neq j)$ is not constant, the existence of Nash equilibrium solutions and an algorithm of construction for them can not be obtained directly as in [1]. But the main idea is still true.

Lemma 5. The pair $\{\bar{t}_1, \bar{t}_2\}$ is a Nash equilibrium solution in the discussed game, if and only if this pair considered as a point of (t_1, t_2)-plane belongs to both graphs $t_1 = t_1^0(t_2 \mid \alpha_1)$ and $t_2 = t_2^0(t_1 \mid \alpha_1)$ presented in the same plane.

The proof of the lemma follows strait away from the definition of a Nash equilibrium solution.

Therefore, to find the solutions one should use Theorem 1 and construct the curves indicated in Lemma 5. If these curves have the common point, these points are the solutions we are looking for. An example of the case with three equilibrium solutions is shown in Fig.3.

The last part of the section we devote to constructing a finite algorithm of searching Nash equilibrium solutions for the case when the best replies function are approximated by piecewise-constant functions. This case is important for practice, where a year is usually taken as the unite of time.

Such an algorithm can also be applied to more general situation when for each participant it is possible not only two, but several variants of benefit rate values, which correspond to different operating modes of gas pipeline. In what follows the set of Nash equilibrium solutions will be denoted by abbreviation NEP.

Let the best reply function of the first participant $t_1^0 = t_1^0(t_2 \mid \alpha_1)$ takes an arbitrary finite number of $t_{1n}^0, n = 1, \ldots, N$ in the corresponding intervals (ζ_{n-1}, ζ_n). At the points $\zeta_n, n = 1, \ldots N-1$ the set $t_1^0(t_2 \mid \alpha_1)$ consists of two points t_{1n}^0 and $t_{1(n+1)}^0$. Similar notations will be used for the function $t_2^0 = t_2^0(t_2 \mid \alpha_2) : t_{2m}^0$, (η_{m-1}, η_m) - for the values and intervals of constancy respectively, $m = 1, \ldots M$. Here $\zeta_0 = \eta_0 = 0, \zeta_N = \eta_M = T$.

Put the points t_{2m}^0 in increasing order. Let us use the previous notations, but remark that the boundaries of the constancy intervals would not be ordered.

Construct an algorithm of determining the set NEP running over the numbers $n = 1, \ldots, N$.

(A1) At the first step of the algorithm with $n = 1$ mark out the points $t_{21}^0, \ldots, t_{2k_1}^0$. If there are no such points go to the next step taking $n+1$ instead of n.

If such points do exist, for each $m = 1, \ldots, k_1$ check the relation

$$t_{1n}^0 \in [\eta_{m-1}, \eta_m]. \qquad (6)$$

If this relation holds true, then a pair $\{t_{1n}^0, t_{2m}^0\}$ belongs to the set NEP.

(A2) For an arbitrary step n one has the value of index k_{n-1} formed at the previous step that corresponds to already considered points t_{2m}^0. If k_{n-1}=M, then the process is finished. When k_{n-1}<M the new points $t_{2m}^0 \in (\zeta_{n-1}, \zeta_n]$, $m \geq k_{n-1}+1$ are marked out, and the new value of k_n is formed. Then for $m = k_{n-1}+1, \ldots, k_n$ one checks the relation (6). The pairs $\{t_{1n}^0, t_{2m}^0\}$, for which it holds true, are referred to the set NEP. If k_n<M unit is added to *n*, and we go to the next step. If $k_n = M$, the process is over.

Theorem 2: The set of Nash equilibrium points NEP in the problem with piecewise-constant best reply functions is determined by the algorithm (A1) - (A2).

VI. Example

In this section the above constructions are concretized by the example, in which the benefit rate functions are linear.

Assume that

$$\varphi_{ij}(t) = a_{ij}t + b_{ij}, \qquad (7)$$

where $0 < a_{i1} < a_{i2}, 0 < b_{i2} < b_{i1}$, $0 \leq t \leq T$, $i, j = 1, 2$. In this case the assumption 1 is provided by inequality

$$a_{i2}T^2 + 2b_{i2}T > 4C_i.$$

The points $\bar{t}_i, \bar{\bar{t}}_i, t_i^*, t_i^{**}$, as well as the function $t_i'(t_j)$ can be expressed by explicit formulas. In particular,

$$t_i^* = \frac{1}{a_{i2}}(-b_{i2} + \sqrt{b_{i2}^{\ 2} + 2a_{i2}C_i});$$

$$\bar{\bar{t}}_i = \frac{1}{a_{i2}}(-b_{i2} + \sqrt{b_{i2}^{\ 2} + 2a_{i2}(a_{i2}T^2 + 2b_{i2}T - 2C_i}).$$

Explicit formulas can be proposed for the functions $\Delta_i(t_1, t_2)$, $f_i(t_1, t_2 \,|\, \alpha_i)$ and for their derivatives. When $0 < \alpha_i < 1$ the expressions for t_i^-, t_i^+ take the form:

$$t_i^- = \frac{1}{a_{i1}}(b_{i1} + (1-\alpha_i)\sqrt{\frac{2a_{i1}C_i}{\alpha_i(2-\alpha_i)}});$$

$$t_i^+ = \frac{1}{a_{i2}}(b_{i2} + (1-\alpha_i)\sqrt{\frac{2a_{i2}C_i}{\alpha_i(2-\alpha_i)}}).$$

The assumption 2 puts more rigorous conditions on parameters of the problem. Let us remark without presenting the corresponding cumbersome inequalities that the obtained system of restrictions is consistent. More than that, it allows to consider in the framework of this approach some real, although approximate data of the planned natural gas pipeline systems.

We finalize the section by presentation some numerical results. Let coefficients in the equality (7) be the same for both participants and are following:

$$a_{ij} = 0.2; \quad b_{i1} = 2; \; b_{i2} = 1.5; \quad i, j = 1.2;$$

as well as parameters $\alpha_i = 0.5$. Put then

$$T = 40, \quad C_i = 60.$$

It is easy to calculate the points $\bar{t}_i, \bar{\bar{t}}_i, t_i^*, t_i^{**}$ for this case. Thus we have t_i^* = 18.12, $\bar{\bar{t}}_i$ =33.20. The functions $\varphi_1(t \,|\, t_2)$ and $\Delta_1(t_1, t_2)$ were shown in Fig.1. The derivative of the function $\Delta_1(t_1, t_2)$ has been shown in Fig. 2.

And finally, in Fig. 3 we present a graphical illustration of the best reply functions and Nash equilibrium points for this case. In presented situation, when both participants have the same parameters the solutions are symmetrical:
(6.64, 4.14), (4.14, 6.64) and (5.03, 5.03).

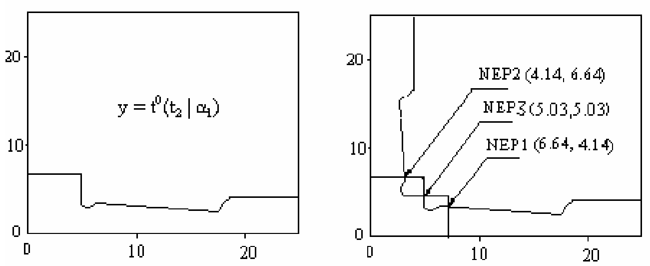

Fig. 3. Best reply function and Nash equilibrium points.

VII. Conclusions

In the paper a game-theoretic problem is considered motivated by investigations related to planning and putting into operation the gas pipeline systems. A new formalization of the problem is proposed that oriented to applications under not purely market price formation mechanism is proposed. In mathematical terms the problem under consideration can be formulated as a non-cooperative two-person game, in which Nash equilibrium solutions are to be found. The best reply functions in such a game are investigated, an algorithm of searching the solutions of the game under some condition is proposed.

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