THE RELIABILITY OF A DRY FRICTION SYSTEM SUBJECTED TO STOCHASTIC FORCING - A NUMERICAL APPROACH

Daniil Iourtchenko
Department of Mathematical Sciences
SPBSPU
Saint-Petersburg, Russia
daniil@phmf.spbstu.ru

Arvid Naess
Department of Mathematical Sciences,
Centre for Ships and Ocean Structures,
NTNU,
Trondheim, Norway
Arvid.Naess@math.ntnu.no

Eirik Mo
Department of Mathematical Sciences,
NTNU,
Trondheim, Norway
mo@ntnu.no

Abstract
The paper presents the reliability analysis of stochastic dry friction system. This system may appear in practice as a result of implementation of the quasi-optimal, bounded in magnitude control law. The path integration method is used to obtain the reliability function, the first passage time and the fatigue characteristics.

Key words
Path integration, outcrossing rate, reliability, fatigue.

1 Introduction

Safety and reliability are extremely important in design of different mechanical systems. A system's reliability may be considered as the probability that no system failure occurs within a given time interval. Often the reliability problem is associated with finding the probability that a system's response stays within a prescribed domain, an outcrossing of which leads to immediate failure. A problem of this type is called *the first passage problem* [Dimentberg, 1988; Lin and Cai 1995; Roberts and Spanos 1990] and it has been extensively studied by a number of authors. The first passage problem is directly related to a solution of the corresponding Pontryagin equation, written with respect to the first excursion time T. Unfortunately, an exact analytical solution to this problem, even for a linear system, is yet to be found. A few strategies have been proposed over the years to deal with this type of problems. One of them is based on an averaging procedure and further problem reformulation for the system's response amplitude or energy. The Markov property of the energy envelope has been used to evaluate the probability of the first passage time for a linear system [Roberts 1976], systems with nonlinear stiffness [Roberts 1978a] or nonlinear damping [Roberts 1978b]. A number of problems have been solved numerically and analytically since then ([Bergman and Heinrich 1982; Spencer and Bergman 1985; Spencer and Bergman 1987; Koyluoglu et.al. 1994]).

This paper is devoted to a reliability investigation of dry friction system. The path integration (PI) code is validated by comparing some results to results of Monte Carlo simulations as well as results, obtained for an equivalent linear system. The latter make sense only for "weak" nonlinearity, i.e. for small values of r ($r << 1$). Its asymptotic analysis was made in [Dimentberg et.al. 2000] with respect to the system's energy. Since the available asymptotic techniques provides reliable estimates for nonlinear systems only in the case of small nonlinearities it is decided to conduct a numerical investigation, comparing some obtained results to the reliability results for an equivalent linear system. The latter is constructed using values of an equivalent viscous damping coefficient and effective frequency. The path integration method has been used earlier for these systems to estimate the stationary response probability density function (PDF) of the state space variables [Iourtchenko et.al. 2006]. Here the PI method has been adapted for obtaining the reliability characteristics of the considered system.

2 Problem Statement and Numerical Approach

2.1 Path Integration approach to reliability

The (SDOF) dynamic systems to be investigated in this paper can all be written in the following form

$$\ddot{X} + g(X, \dot{X}) = \xi(t), \tag{1}$$

where $g(\cdot,\cdot)$ denotes a function to be specified in each particular case, while $\xi(t)$ throughout denotes a zero-mean, stationary Gaussian white noise process satisfying $\mathrm{E}[\,\xi(t)\,\xi(t+\tau)\,] = D\delta(\tau)$ for a positive intensity parameter D. Application of the external quasi-optimal control policy leads to a dry friction law, whereas application of the quasi-optimal control force results in parametrically controlled systems with a jumpwise variation of either the systems's stiffness, moment of inertia or both. The latter happens through a variation of the pendulum's length and such a system is well known as a swing.

Equation (1) will be construed as an Itô stochastic differential equation (SDE), that is,

$$dZ(t) = h(Z(t))\,dt + b\,dB(t), \tag{2}$$

where the state space vector process $Z(t) = (X(t), Y(t))^T = (X(t), \dot{X}(t))^T$ has been introduced; $h = (h_1, h_2)^T$ with $h_1(Z) = Y$ and $h_2(Z) = -g(X, Y)$; $b = (0, \sqrt{D})^T$, and $B(t)$ denotes a standard Brownian motion process. From Eq. (2) it follows immediately that $Z(t)$ is a Markov process, and it is precisely the Markov property that will be used in the formulation of the PI procedure.

The reliability is defined in terms of the displacement response process $X(t)$ in the following manner, assuming that all events are well defined,

$$\begin{aligned} R(\,T\,|\,x_0, 0, t_0) = \mathrm{Prob}\{x_l < X(t) < x_c;\\ t_0 < t \le T | X(t_0) = x_0,\, Y(t_0) = 0\}, \end{aligned} \tag{3}$$

where x_l, x_c are the lower and upper threshold levels defining the safe domain of operation. Hence the reliability $R(T|x_0, 0, t_0)$, as we have defined it here, is the probability that the system response $X(t)$ stays above the threshold x_l and below the threshold x_c throughout the time interval (t_0, T) given that it starts at time t_0 from x_0 with zero velocity ($x_l < x_0 < x_c$). In general, it is impossible to calculate the reliability exactly as it has been specified here since it is defined by its state in continuous time, and for most systems the reliability has to be calculated numerically, which inevitably will introduce a discretization of the time. Assuming that the realizations of the response process $X(t)$ are piecewise differentiable with bounded slope with probability one, the following approximation is introduced

$$\begin{aligned} R(\,T\,|\,x_0, 0, t_0) \approx \mathrm{Prob}\{x_l < X(t_j) < x_c,\\ j = 1, \ldots, n | X(t_0) = x_0,\, Y(t_0) = 0\}, \end{aligned} \tag{4}$$

where $t_j = t_0 + j\Delta t$, $j = 1, \ldots, n$, and $\Delta t = (T - t_0)/n$. With the assumptions made, the rhs of this equation can be made to approximate the reliability as closely as desired by appropriately choosing Δt, or, equivalently, n. Within the adopted approximation, it is realized that the reliability can now be expressed in terms of the joint conditional PDF $f_{X(t_1)\ldots X(t_n)|X(t_0),Y(t_0)}(\cdot,\ldots,\cdot|x_0, 0)$ as follows, which is just a rephrasing of Eq. (4),

$$\begin{aligned} &R(\,T\,|\,x_0, 0, t_0) \approx \\ &\int_{x_l}^{x_c} \cdots \int_{x_l}^{x_c} f_{(\cdots)}(x_1, \ldots, x_n | x_0, 0)\, dx_1 \cdots dx_n. \end{aligned} \tag{5}$$

Due to the Markov property of the state space vector process $Z(t) = (X(t), Y(t))^T$, we may express the joint PDF of $Z(t_1), \ldots, Z(t_n)$ in terms of the transition probability density function

$$\begin{aligned} &p(z, t|z', t') = f_{Z(t)|Z(t')}(z|z') = \\ &f_{Z(t)Z(t')}(z, z')/f_{Z(t')}(z'), \quad (f_{Z(t')}(z') \neq 0) \end{aligned}$$

in the following way

$$\begin{aligned} &f_{Z(t_1)\ldots Z(t_n)|Z(t_0)}(z_1, \ldots, z_n|z_0) = \\ &\prod_{j=1}^{n} p(z_j, t_j|z_{j-1}, t_{j-1}). \end{aligned} \tag{6}$$

This leads to the expression ($z_0 = (x_0, 0)^T$, $dz_j = dx_j\,dy_j$, $j = 1, \ldots, n$)

$$\begin{aligned} &R(\,T\,|\,x_0, 0, t_0) \approx \int_{-\infty}^{\infty}\int_{x_l}^{x_c} \cdots \int_{-\infty}^{\infty}\int_{x_l}^{x_c} \prod_{j=1}^{n} [\\ &p(z_j, t_j|z_{j-1}, t_{j-1})]\, dz_1 \cdots dz_n, \end{aligned} \tag{7}$$

which is the path integration formulation of the reliability problem. The numerical calculation of the reliability is done iteratively in an entirely analogous way as in standard path integration. To show that, let us introduce a reliability density function (RDF) $q(z, t|z_0, t_0)$ as follows,

$$\begin{aligned} &q(z_2, t_2|z_0, t_0) = \\ &\int_{-\infty}^{\infty}\int_{x_l}^{x_c} p(z_2, t_2|z_1, t_1)\, p(z_1, t_1|z_0, t_0)\, dz_1, \end{aligned} \tag{8}$$

and $(n > 2)$

$$\begin{aligned} &q(z_k, t_k|z_0, t_0) = \\ &\int_{-\infty}^{\infty}\int_{x_l}^{x_c} p(z_k, t_k|z_{k-1}, t_{k-1})\bullet \\ &q(z_{k-1}, t_{k-1}|z_0, t_0)\, dz_{k-1}, \;\; k = 3, \ldots, n. \end{aligned} \tag{9}$$

The reliability is then finally calculated approximately as ($T = t_n$)

$$R(T\,|\,x_0, 0, t_0) \approx \int_{-\infty}^{\infty}\int_{x_l}^{x_c} q(z_n, t_n|z_0, t_0)\, dz_n. \tag{10}$$

The complementary probability distribution of the time to failure T_e, i.e. the first passage time, is given by the reliability function. The mean time to failure $\langle\, T_e\,\rangle$ can thus be calculated by the equation

$$\langle\, T_e\,\rangle = \int_0^{\infty} R(\,\tau\,|\,x_0, 0, t_0)\, d\tau \tag{11}$$

To evaluate the reliability function it is required to know the transition probability density function $p(z, t|z', t')$, which is unknown for the considered nonlinear systems. However, from Eq. (2) it is seen that for small $t - t'$ it can be determined approximately, which is what is needed for the numerical calculation of the reliability. A detailed discussion of this, and the iterative integrations of Eqs. (8) and (9), is given in [Iourtchenko et.al. 2006; Naess et.al. 2007]. Concerning the integrations, there is, however, one small difference between the present formulation and that described in these references. In Eqs. (8) and (9), the integration in the x-variable only extends over the interval (x_l, x_c). The infinite upper and lower limits on the y-variable are replaced by suitable constants determined by e.g. an initial Monte Carlo simulation.

If the system response $Z(t)$ has a stationary response PDF $f_Z(z)$ as $t \to \infty$, it follows that the conditional response PDF $f_{\{Z(t_n)|Z(0),\, x_l<X(t_j)<x_c;\, 0\le j\le n-1\}}(z)$ also reaches a stationary density, say $q^*(z)$, when $t_n \to \infty$. This means that the reliability process eventually becomes memoryless, and hence the RDF converges $q(z, t_n|z_0, t_0) \to q^*(z)Ke^{-\nu t_n}$ for some constants K and ν as $t_n \to \infty$. Also the numerical method should reach stationarity in the conditional density. This also implies that the numerically estimated reliability function must be exponential, since the same relative amount of probability mass leaves the system at every iteration. So in the end, the only thing one should need for a good reliability estimate is the behavior in the transient phase, and the exponential decay thereafter.

2.2 General comments about the numerical procedure

The numerical calculations were performed for a 256×256 mesh in the state space, with very high grid resolution around the axes for the inertia controlled system and swing system, because the PDFs have discontinuities along the axes and high spikes at the discontinuity that requires a well adapted spline representation [Iourtchenko et.al. 2006]. More specifically, the grid resolution was determined by an exponentially decaying function away from each coordinate axis. Because of the discontinuities, there are no grid points on the axes themselves. However, the interpolant will be globally smooth and assume finite values also on the axes. Hence, there is no true discontinuity in the interpolant even if the gradients of the interpolant may be very large at the axes. The time step was 0.01 for all simulations, and the noise intensity D was set to 1.0. The initial choice of time step is determined by the characteristic time constants of the dynamic system, which can be either seen from the system equations or from a short Monte Carlo simulation of the dynamic response of the system. As is typically done for verifying the convergence of numerical solutions, the accuracy of the calculated PI solution may be checked by changing repeatedly, if required, the size of the time step, for example by a factor 2.

For all simulations, the reliability was computed using the barriers $x_c = 2.5\sigma_x$, $x_c = 3.0\sigma_x$, and $x_c = 3.5\sigma_x$. The lower barrier is either $x_l = -\infty$, one-sided barrier case, or $x_l = -x_c$ for two-sided reliability. These bounds were far enough out in the tails that interpolation of the RDF from equations (8) and (9) was no problem.

It should be mentioned that for the system studied in this paper, the calculated reliability function displayed a distinctive exponential behavior asymptotically, as one would expect. That is, after some transient time, the reliability function could not be distinguished from a straight line when plotted on a logarithmic scale. In addition, the PDF for the time to failure has a right tail that is exponential with the same exponent, which again is verified by plotting the PDF on a logarithmic scale. The oscillatory behaviour of the PDFs of the time to failure, as seen on the close ups, largely reflects the transient dynamics of the systems due to initial conditions.

2.3 Monte Carlo simulation

To check the numerical results, Monte-Carlo simulations (MCS) have been run for a few selected cases. A main problem is that the probability of crossing a high reliability level is small, so the simulation will have to run for a long time before this happens. Since a good approximation of the PDF for the first passage time needs a large number of Monte-Carlo simulations, this easily becomes a very time consuming method. The verification of the numerical results by Monte Carlo simulations are therefore carried out on two levels. First, the expected first passage time is estimated directly from simulated response time histories for the lowest level (= 2.5 σ), where σ equals the standard deviation of stationary response. For all the models investigated in this paper, the estimated expected first passage time obtained by MCS agreed with the corresponding one calculated by PI within the accuracy of the MCS estimate, that is, within a few per cent.

It is important to notice that estimating the full PDF, and here especially the transient behavior, is very time consuming with Monte Carlo methods without a parametric model. Path integration, however, calculates this directly, and if only the transient behavior is needed, the PDF can be found with high accuracy with a fairly short simulation.

When comparing the results for the MC and PI methods, one should remember that the strengths and weaknesses of the numerical methods are also very different. The main problem for the PI method, is that the PDFs may have sharp discontinuities or peaks that makes the interpolation difficult.

3 Results for a stochastic dry friction system

In this section some derivations made in [Dimentberg et.al. 2000] for a stochastic system with dry friction are briefly discussed. It is worth mentioning that for the parametric systems the stochastic averaging procedure results in an exponential response PDF for response energy, whereas the dry friction system has an exponent in power of the square root of the response energy. Therefore, for the parametrically controlled systems, the case of small nonlinearity cannot be caught by the averaging procedure and needs to be investigated numerically. For the system with dry friction it is possible to use an approximate analysis for a small value of the dry friction coefficient. Early results on the use of PI for an oscillator with dry friction are reported in [Naess and Johnsen 1993].

3.1 The first passage time

Consider the following nonlinear system, subjected to the zero mean, stationary Gaussian white noise $\xi(t)$ introduced above:

$$\begin{gathered}\ddot{X} + r\,\mathrm{sign}(\dot{X}) + \Omega^2 X = \xi(t), \quad 0 \le t \le t_f. \\ < \xi(t)\xi(t+\tau) >= D\delta(\tau)\end{gathered} \tag{12}$$

Applying the stochastic averaging procedure and following the derivations made in [Dimentberg et.al. 2000] the first passage time may be found as:

$$\begin{aligned}&T(c) = \frac{[\,\mathrm{Ei}(2\lambda\sqrt{\bar{c}}) - \mathrm{Ei}(2\lambda\sqrt{c})\,]}{2\Omega\lambda^2} - \frac{\sqrt{\bar{c}} - \sqrt{c}}{\Omega\lambda} - \\ &\frac{\ln(\bar{c}/c)}{4\Omega\lambda^2} \\ &c = \frac{E}{D/4\Omega}, \quad \bar{c} = \frac{\bar{E}}{D/4\Omega}, \\ &\lambda = \frac{2\sqrt{2}\mu}{\pi}, \quad \mu = \frac{r}{\sqrt{D\Omega}}.\end{aligned} \tag{13}$$

where $\mathrm{Ei}(y)$ is an exponential integral function, D is a noise intensity, $\bar{E}$ is a critical value of energy. Thus, an analytical expression (13) may be used for reliability estimates, keeping in mind that r should be small. This result may be compared to one, reported in [Lin and Cai 1995], keeping in mind that the value of an equivalent viscous damping coefficient is equal to:

$$\alpha_{\mathrm{eq}}^{\mathrm{df}} = \frac{16r^2}{3\pi^2 D}.$$

It can be seen from the comparison with the result for the linear system [Lin and Cai 1995] that expression (13) has an additional second term, which is nonnegative. Moreover, the exponential integral function (13) depends on the square root of the system's energy, whereas the formula for an equivalent linear system [Lin and Cai 1995] predicts dependence on the system's energy itself. Both these facts indicates that the first passage time to failure for the dry friction system should be less than that for an equivalent linear system.

Numerical simulations, conducted using the PI method, have shown that the joint response pdf has a single peak, at small values of r, which splits into two peaks, moving away from each other, when the nonlinearity parameter r increases [Iourtchenko et.al.]. A peak of the probability density of time to failure moves left when value of r increases, which indicates deterioration of the system's reliability. Figure 1 and figure 2 present the reliability function for $r = 0.15$ and $r = 0.25$ correspondingly for different values of the crossing level $p = x_c/\sigma_x$. These results show strong dependence of the reliability function on r, i.e. an increase of r increases the slope of the reliability function, consequently decreasing the time to failure. At first glance, this may seem odd, but remember that an increase in r leads to a decrease in σ_x, and therefore in the critical level. On the other hand, an increase of the crossing level leads, as expected, to an increase of the first passage time value for a fixed value of r.

Table 1 presents results of numerical simulations for the first passage time. Data in Table 1 have been compared to the data obtained for an equivalent linear system. Direct comparison of these results, for the same level of energy dissipation in both systems, showed that the dry friction system has significantly (at least twice) smaller value of failure time than that of the equivalent linear system, which indicates a relatively poor reliability of the dry friction system.

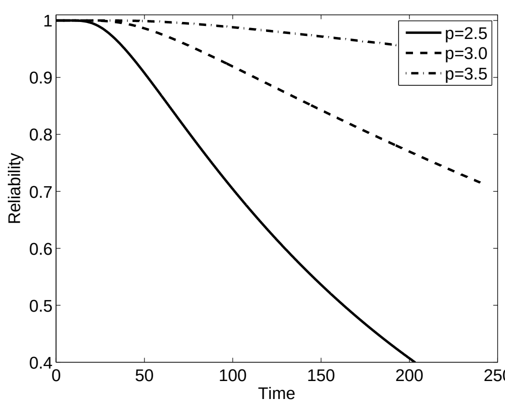

Figure 1. Reliability function of the dry friction system for $r = 0.15$.

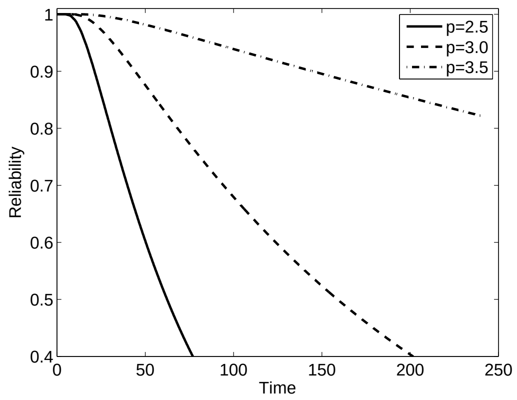

Figure 2. Reliability function of the dry friction system for $r = 0.25$.

3.2 The fatigue life

It is known [Bolotin, 1984] that the fatigue life or expected time for a system to work properly before failure due to fatigue occurs may be found as:

$$\frac{1}{T_*} = \int \frac{p(y)dy}{T_e(y)N_s(y)} \tag{14}$$

where $p(y)$ - the response pdf of crossing, $T_e(y)$ - the effective period of the process, e.g. $T_e(y) = 2\pi/\omega_e$ and

$$N_s(y) = \begin{cases} N_0(q/y)^m & (y \geq q) \\ \infty & (y < q) \end{cases} \tag{15}$$

Here q is the threshold value. In the case, when $T_e(y)$ is a constant, the formula (14) may be written as:

$$\frac{T_0}{T_*} = \int \left(\frac{y}{q}\right)^m p(y)dy, \;\; T_0 = T_e N_0 \tag{16}$$

The required pdf of crossing has been obtained by the PI method, and used to calculate T_0/T_* value for different values of m and q. For the purposes of calculations the following values of parameters were taken: $T_0 = 1$, $D = 1$. These results are presented in Fig.3, Fig.4, Fig.5 for values of $r = 0.15$, $r = 0.20$, $r = 0.25$ correspondingly.

p	2.5σ	3.0σ	3.5σ
r			
0.15	0.2167	0.6009	2.6885
0.20	0.1344	0.3697	1.7757
0.25	0.0814	0.2169	1.0834

Table 1. Expected time to upcrossing for the dry friction system. All numbers to be $\times 10^3$.

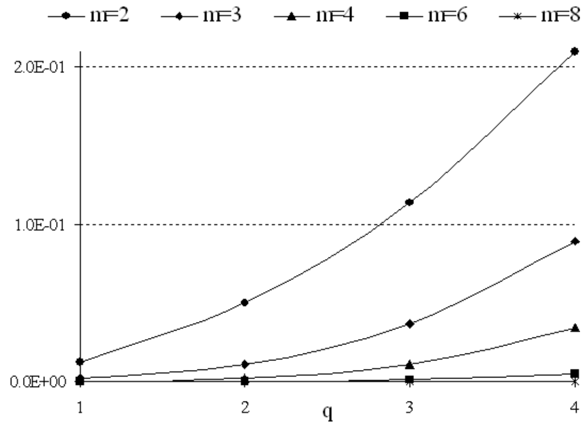

Figure 3. Dimensionless fatigue life T_0/T_* for $r = 0.15$.

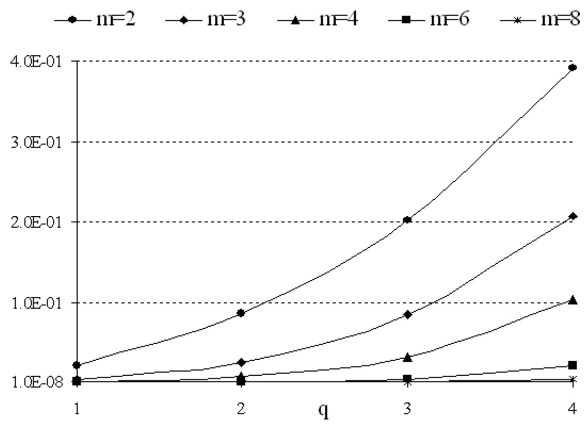

Figure 4. Dimensionless fatigue life T_0/T_* for $r = 0.20$.

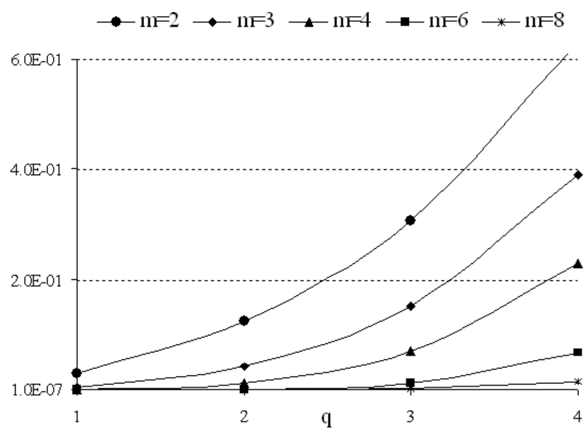

Figure 5. Dimensionless fatigue life T_0/T_* for $r = 0.25$.

These results indicate that the increase of nonlinearity leads to decrease of fatigue life for given values of m and q. It also can bees seen that higher values of m provides longer fatigue life.

4 Conclusions

In the paper the authors have considered a reliability problems for strongly nonlinear dry friction stochastic systems. The numerical results presented in the paper are obtained by the path integration method, which was adjusted to handle reliability problems. The results were verified by Monte-Carlo simulations and the results obtained by the path integration method for an equivalent linear system. Generated results demonstrated that the reliability of the system strongly depends on the nonlinearity parameter r, especially for low values of the upcrossing level. It has been found that the dry friction system or the system with an external, bounded in magnitude control law, has poor reliability compared to its equivalent linear system, although it is capable of reducing the system's response energy. Finally, the numerical investigation of the fatigue life has been conducted.

Acknowledgements

Author Iourtchenko D.V. was supported by the Council of the Russian Federation President (MD-6147.2008.1) for young scientists. This support is most highly appreciated.

References

Bergman L.A., Heinrich J.C. (1982). On the reliability of the linear oscillator and systems of coupled oscillators. *Int. J. for Numerical Methods in Engineering*. Vol. 18, pp. 1271-1295.

Bolotin V.V. (1984). *Random Vibrations of Elastic Systems* . Kluwer, USA.

Chen J.B., Li J. (2005). Dynamic response and reliability analysis of non-linear stochastic structures. *Probabilistic Engineering Mechanics*. Vol. 20, pp. 33-44.

Dimentberg M.F. (1988). *Statistical Dynamics of Nonlinear and Time-Varying Systems*. Research Studies Press, Taunton, UK.

Dimentberg M.F., Iourtchenko D.V., Bratus A. (2000). Optimal bounded control of steady-state random vibrations. *Probabilistic Engineering Mechanics*. Vol. 15, pp. 381-386.

Iourtchenko D.V. (2000) Stochastic optimal bounded control for a system with the Boltz cost function. *J. Vibration and Control*. Vol. 6, pp. 1195-1204.

Iourtchenko D.V., Dimentberg M.F. (2001a). Energy balance for random vibrations of piecewise-conservative systems. *Journal of Sound and Vibration.* **248**, N 5, – P. 913 - 923.

Iourtchenko D.V., Dimentberg M.F. (2001b). Energy balance for random vibrations of piecewise - conservative systems. *Journal of Sound and Vibration*. Vol. 248, No. 5, pp. 913-923.

Iourtchenko D.V., Mo E., Naess A. (2006) Response probability density functions of strongly nonlinear systems by the path integration method. *Int. J. of Non-Linear Mechanics*. Vol. 41, No. 5, pp. 693-705.

Iourtchenko D.V., Mo E., Naess A. Reliability of strongly nonlinear SDOF dynamic systems by the path integration method. To appear in *Journal of Applied Mechanics*.

Johnsen, J. M. (1992). *Response Statistics of Nonlinear Dynamic Systems*. PhD Thesis, Department of Structural Engineering, NTH, Trondheim, 1992.

Koyluoglu H.U., Nielsen S.R.K., Iwankiewicz R. (1994). Reliability of non-linear oscillators subjected to Poisson driven impulses. *Journal of Sound and Vibraion*. Vol. 176, No. 1, pp. 19-33.

Lin, Y. K. and Cai, G. Q. (1995). *Probabilistic Structural Dynamics*. McGraw-Hill. New York.

Naess, A., Kolnes, F.E., Mo, E. (2007) Stochastic spur gear dynamics by numerical path integration. *J. of Sound and Vibration*. Vol. 302, pp. 936-950.

Naess, A., Johnsen J. M. (1993). The effect of Coulomb damping on the response statistics of a SDOF oscillator subjected to white noise excitation. *Proceedings 2nd European Conference on Structural Dynamics (Eurodyn '93)*, Trondheim, Norway, June 1993.

Risken H. (1989). *The Fokker-Plank equstion: methods of solutions and applications*. Springer-Verlag, New York, USA.

Roberts, J.B. (1976). First passage time for the envelope of a randomly excited linear oscillator. *Journal of Sound and Vibraion*. Vol. 46, No. 1, pp. 1-14.

Roberts, J.B. (1978a). First passage time for oscillators with non-linear restoring forces. *Journal of Sound and Vibraion*. Vol. 56, No. 1, pp. 71-86.

Roberts, J.B. (1978b). First passage time for oscillators with nonlinear dampimg. *ASME Journal of Applied Mechanics*. Vol. 45, pp. 175-180.

Roberts, J.B. and Spanos, P.D. (1990). *Random Vibration and Statistical Linearization*.Wiley, New York.

Spencer Jr. B.F., Bergman L.A. (1985). On the reliability of a simple hysteretic system. *J. of Engineering Mechanics*. Vol. 111, No. 12, pp. 1502-1513.

Spencer Jr. B.F., Bergman L.A. (1987). First passage time for linear systems with stochastic coefficients. *Probabilistic Engineering Mechanics*. Vol. 2, No. 1, pp. 46-53.