Complete Dynamic Model of a Stewart version Parallel Manipulator

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1 Introduction

The dynamic model of a parallel manipulator in free space can be mathematically represented, in Cartesian space, by a system of nonlinear differential equations:

$$\mathbf{I}(\mathbf{x})\cdot\ddot{\mathbf{x}}+\mathbf{V}(\mathbf{x},\dot{\mathbf{x}})\cdot\dot{\mathbf{x}}+\mathbf{G}(\mathbf{x})=\mathbf{f} \quad (1)$$

$\mathbf{I}(\mathbf{x})$ being the inertia matrix, $\mathbf{V}(\mathbf{x},\dot{\mathbf{x}})$ the Coriolis and centripetal terms matrix, $\mathbf{G}(\mathbf{x})$ a vector of gravitational generalized forces, $\mathbf{x}$ the generalized position of the moving platform (or end-effector) and $\mathbf{f}$ the controlled generalized force applied on the end-effector. Thus,

$$\mathbf{f}=\mathbf{J}^{T}(\mathbf{x})\cdot\boldsymbol{\tau} \quad (2)$$

where $\boldsymbol{\tau}$ is the generalized force developed by the actuators and $\mathbf{J}(\mathbf{x})$ is a jacobian matrix.

The dynamic model of a parallel manipulator is usually developed following one of two approaches: the Newton-Euler or the Lagrange methods. Do and Yang [1], and Reboulet and Berthomieu [2] use this method on the dynamic modeling of a Stewart platform. Ji [3] presents a study on the influence of leg inertia on the dynamic model of a Stewart platform. Dasgupta and Mruthyunjaya [4] used the Newton-Euler approach to develop a dynamic model of the Stewart platform. This method was also used by Khalil and Ibrahim [5], Riebe and Ulbrich [6], Guo and Li [7], and Carvalho and Ceccarelli [8], among others.

The Lagrange method was used by Nguyen and Pooran [9] to model a Stewart platform, modeling the legs as point masses. Liu et al. [10], and Lebret et al. [11] follow a similar approach, but trying to increase the physical meaning of the obtained mathematical expressions. Geng et al. [12] used the Lagrange's method to develop the equations of motion for a class of Stewart platforms. Some simplifying assumptions regarding the manipulator geometry and inertia distribution were considered. Lagrange's method was also used by Gregório and Parenti-Castelli [13], and Caccavale et al. [14], for example.

Alternative methods have also been searched, trying to reduce computational load, namely the ones based on the principle of virtual work [15-16], and screw theory [17].

In this paper the Lagrange's formulation is used in the complete dynamic modeling of a 6-dof parallel manipulator. The involved computational effort is evaluated and compared with the one presented by a proposed simplified model. It is shown the proposed simplified model presents a much lower computational burden, being representative of the mechanical behavior of the manipulator.

2 Manipulator Kinematic Structure

The manipulator structure comprises a fixed (base) platform and a moving (payload) platform, linked together by six independent, identical, open kinematic chains (Figure 1). Each chain comprises two links: the first link (linear actuator) is always normal to the base and has a variable length, l_i, with one of its ends fixed to the base and the other one attached, by a universal joint, to the second link; the second link (fixed-length link) has a fixed length, L, and is attached to the payload platform by a spherical joint. Points B_i and P_i are the connecting points to the base and payload platforms. They are located at the vertices of two semi-regular hexagons, inscribed in circumferences of radius r_B and r_P, that are coplanar with the base and payload platforms.

For kinematic modeling purposes, two frames, {P} and {B}, are attached to the centre of mass of the moving and base platforms, respectively.

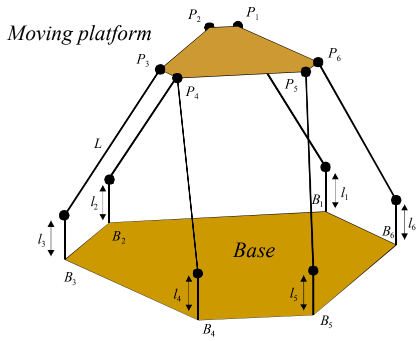

Fig. 1. Manipulator kinematic structure.

The generalized position of frame {P} relative to frame {B} may be represented by the vector:

$$\begin{aligned}{}^{B}\mathbf{x}_{P}|_{B|E} &= \begin{bmatrix}x_P & y_P & z_P & \psi_P & \theta_P & \varphi_P\end{bmatrix}^T\\ &= \begin{bmatrix}{}^{B}\mathbf{x}^{T}_{P(pos)}|_{B} & {}^{B}\mathbf{x}^{T}_{P(o)}|_{E}\end{bmatrix}^T\end{aligned} \quad (3)$$

where ${}^{B}\mathbf{x}_{P(pos)}|_{B}=\begin{bmatrix}x_P & y_P & z_P\end{bmatrix}^T$ is the position of the origin of frame {P} relative to frame {B}, and ${}^{B}\mathbf{x}_{P(o)}|_{E}=\begin{bmatrix}\psi_P & \theta_P & \varphi_P\end{bmatrix}^T$ defines an Euler angle system representing orientation of frame {P} relative to {B}. The rotation matrix is given by:

$${}^{B}\mathbf{R}_{P}=\begin{bmatrix}C\psi_P C\theta_P & C\psi_P S\theta_P S\varphi_P - S\psi_P C\varphi_P & C\psi_P S\theta_P C\varphi_P + S\psi_P S\varphi_P\\ S\psi_P C\theta_P & S\psi_P S\theta_P S\varphi_P + C\psi_P C\varphi_P & S\psi_P S\theta_P C\varphi_P - C\psi_P S\varphi_P\\ -S\theta_P & C\theta_P S\varphi_P & C\theta_P C\varphi_P\end{bmatrix} \quad (4)$$

$S(\cdot)$ and $C(\cdot)$ correspond to the sine and cosine functions, respectively.

The manipulator position and velocity kinematic models are well known, being obtainable from the geometrical analysis of the kinematics chains. The velocity kinematics is represented by the Euler angles jacobian matrix, $\mathbf{J}_E$, or the kinematic jacobian, $\mathbf{J}_C$. These jacobians relate the velocities of the active joints, the actuators, to the generalized velocity of the moving platform. Therefore,

$$\dot{\mathbf{l}}=\mathbf{J}_E\cdot{}^{B}\dot{\mathbf{x}}_{P}|_{B|E}=\mathbf{J}_E\cdot\begin{bmatrix}{}^{B}\dot{\mathbf{x}}_{P(pos)}|_{B}\\ {}^{B}\dot{\mathbf{x}}_{P(o)}|_{E}\end{bmatrix} \quad (5)$$

$$\dot{\mathbf{l}}=\mathbf{J}_C\cdot{}^{B}\dot{\mathbf{x}}_{P}|_{B}=\mathbf{J}_C\cdot\begin{bmatrix}{}^{B}\dot{\mathbf{x}}_{P(pos)}|_{B}\\ {}^{B}\boldsymbol{\omega}_{P}|_{B}\end{bmatrix} \quad (6)$$

$$\dot{\mathbf{l}}=\begin{bmatrix}\dot{l}_1 & \dot{l}_2 & \cdots & \dot{l}_6\end{bmatrix}^T \quad (7)$$

$${}^{B}\boldsymbol{\omega}_{P}|_{B}=\mathbf{J}_A\cdot{}^{B}\dot{\mathbf{x}}_{P(o)}|_{E} \quad (8)$$

$$\mathbf{J}_A=\begin{bmatrix}0 & -S\psi_P & C\theta_P C\psi_P\\ 0 & C\psi_P & C\theta_P S\psi_P\\ 1 & 0 & -S\theta_P\end{bmatrix} \quad (9)$$

Vectors ${}^{B}\dot{\mathbf{x}}_{P(pos)}|_{B}\equiv{}^{B}\mathbf{v}_{P}|_{B}$ and ${}^{B}\boldsymbol{\omega}_{P}|_{B}$ represent, the linear and angular velocity of the moving platform, relative to {B}, and

${}^{B}\dot{\mathbf{x}}_{P(o)}|_E$ represents the Euler angles time derivative.

3 Manipulator Dynamic Modeling

Generally speaking, the dynamic model of a mechanical system may be obtained by means of the well known Lagrange equation:

$$\frac{d}{dt}\left(\frac{\partial K(\mathbf{h},\dot{\mathbf{h}})}{\partial \dot{\mathbf{h}}}\right)-\frac{\partial K(\mathbf{h},\dot{\mathbf{h}})}{\partial \mathbf{h}}+\frac{\partial P(\mathbf{h})}{\partial \mathbf{h}}=\boldsymbol{\zeta} \tag{10}$$

where K and P are the system total kinetic and potential energies, $\mathbf{h}$ is a vector of generalized coordinates, and $\boldsymbol{\zeta}$ represents the generalized force applied to the system. Vectors $\mathbf{h}$ and $\boldsymbol{\zeta}$ are expressed in the same referential.

Considering the parallel manipulator, and expressing the Lagrange equation as a function of the moving platform generalized position, ${}^{B}\mathbf{x}_P|_{B|E}$, results in:

$$\frac{d}{dt}\left(\frac{\partial K\left({}^{B}\mathbf{x}_P|_{B|E},{}^{B}\dot{\mathbf{x}}_P|_{B|E}\right)}{\partial {}^{B}\dot{\mathbf{x}}_P|_{B|E}}\right)-\frac{\partial K\left({}^{B}\mathbf{x}_P|_{B|E},{}^{B}\dot{\mathbf{x}}_P|_{B|E}\right)}{\partial {}^{B}\mathbf{x}_P|_{B|E}}+\frac{\partial P\left({}^{B}\mathbf{x}_P|_{B|E}\right)}{\partial {}^{B}\mathbf{x}_P|_{B|E}}={}^{P}\mathbf{f}|_{B|E} \tag{11}$$

where ${}^{P}\mathbf{f}|_{B|E}^{T}$ represents the generalized force acting on the centre of mass of the moving platform. This vector may be written as:

$${}^{P}\mathbf{f}|_{B|E}=\left[{}^{P}\mathbf{F}^{T}|_B \quad {}^{P}\mathbf{M}^{T}|_E\right]^{T} \tag{12}$$

Vector ${}^{P}\mathbf{F}|_B$ represents the total force acting on the centre of mass of the moving platform, expressed in the base frame, {B}, and vector ${}^{P}\mathbf{M}|_E$ represents the total moment acting on the moving platform, expressed using the Euler angles system. Thus, this representation does not allow a clear physical interpretation of ${}^{P}\mathbf{M}|_E$.

In order to simplify the used language, ${}^{P}\mathbf{f}|_{B|E}$ will be referred as the generalized force vector acting on the centre of mass of the moving platform expressed using the Euler angles system. In a similar way, ${}^{P}\mathbf{f}|_B$ will be referred as the generalized force vector acting on the centre of mass of the moving platform expressed in the base frame {B}. That is,

$${}^{P}\mathbf{f}|_{B}=\left[{}^{P}\mathbf{F}^{T}|_B \quad {}^{P}\mathbf{M}^{T}|_B\right]^{T} \tag{13}$$

where ${}^{P}\mathbf{M}|_B=\mathbf{J}_A^{-T}\cdot{}^{P}\mathbf{M}|_E$ represents the moment vector applied in the mobile platform and expressed in the base frame {B}. The corresponding actuating forces, $\boldsymbol{\tau}$, may be obtained using the relation

$$\boldsymbol{\tau}=\mathbf{J}_E^{-T}\cdot{}^{P}\mathbf{f}|_{B|E} \tag{14}$$

3.1 Manipulator Kinetic Energy

The total kinetic energy, K, may be computed as the sum of the kinetic energies of all the rigid bodies: moving platform, 6 actuators and 6 fixed-length links.

$$K=K_P+\sum_{i=1}^{6}K_{A_i}+\sum_{i=1}^{6}K_{L_i} \tag{15}$$

where K_P, K_{A_i} and K_{L_i} represent moving platform, actuator, and fixed-length link kinetic energies, respectively.

3.1.1 Moving Platform Kinetic Energy

The moving platform kinetic energy may be computed as the sum of two components: $K_{P(tra)}$ being the translational kinetic energy and $K_{P(rot)}$ being the rotational kinetic energy.

$$K_P=K_{P(tra)}+K_{P(rot)} \tag{16}$$

The translational kinetic energy may be easily computed using the following equation:

$$K_{P(tra)}=\frac{1}{2}\cdot{}^{B}\mathbf{v}_P^{T}|_B\cdot\mathbf{I}_{P(tra)}\cdot{}^{B}\mathbf{v}_P|_B \tag{17}$$

where $I_{P(tra)}$ is the translational inertia matrix of the moving platform and m_P is its mass:

$$\mathbf{I}_{P(tra)}=\mathrm{diag}\left(\begin{bmatrix}m_P & m_P & m_P\end{bmatrix}\right) \tag{18}$$

In a similar way, the rotational kinetic energy may be easily computed by the equation:

$$K_{P(rot)}=\frac{1}{2}\cdot{}^{B}\boldsymbol{\omega}_P^{T}|_B\cdot\mathbf{I}_{P(rot)}|_B\cdot{}^{B}\boldsymbol{\omega}_P|_B \tag{19}$$

$\mathbf{I}_{P(rot)}|_B$ representing the rotational inertia matrix, expressed in the base frame {B}. This matrix can be written as a function of the rotational inertia matrix expressed in the mobile platform frame {P}:

$$\mathbf{I}_{P(rot)}|_B={}^{B}\mathbf{R}_P\cdot\mathbf{I}_{P(rot)}|_P\cdot{}^{B}\mathbf{R}_P^{T} \tag{20}$$

$$\mathbf{I}_{P(rot)}|_P=\mathrm{diag}\left(\begin{bmatrix}I_{P_{xx}} & I_{P_{yy}} & I_{P_{zz}}\end{bmatrix}\right) \tag{21}$$

Adding the translational and rotational components results in the total moving platform kinetic energy:

$$K_P=\frac{1}{2}\cdot\begin{bmatrix}{}^{B}\mathbf{v}_P|_B\\ {}^{B}\boldsymbol{\omega}_P|_B\end{bmatrix}^{T}\cdot\begin{bmatrix}\mathbf{I}_{P(tra)} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{P(rot)}|_B\end{bmatrix}\cdot\begin{bmatrix}{}^{B}\mathbf{v}_P|_B\\ {}^{B}\boldsymbol{\omega}_P|_B\end{bmatrix} \tag{22}$$

where

$$\mathbf{I}_P|_B=\begin{bmatrix}\mathbf{I}_{P(tra)} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{P(rot)}|_B\end{bmatrix} \tag{23}$$

is the moving platform inertia matrix expressed in frame {B}.

On the other hand, using equations (8) and (19):

$$K_{P(rot)}=\frac{1}{2}\cdot{}^{B}\dot{\mathbf{x}}_{P(o)}^{T}|_E\cdot\mathbf{J}_A^{T}\cdot\mathbf{I}_{P(rot)}|_B\cdot\mathbf{J}_A\cdot{}^{B}\dot{\mathbf{x}}_{P(o)}|_E \tag{24}$$

therefore,

$$\mathbf{I}_{P(rot)}|_E=\mathbf{J}_A^{T}\cdot{}^{B}\mathbf{R}_P\cdot\mathbf{I}_{P(rot)}|_P\cdot{}^{B}\mathbf{R}_P^{T}\cdot\mathbf{J}_A \tag{25}$$

is the rotational moving platform inertia matrix, expressed using the Euler angles system.

The total moving platform kinetic energy may be rewritten as:

$$K_P=\frac{1}{2}\cdot\begin{bmatrix}{}^{B}\dot{\mathbf{x}}_{P(pos)}|_B\\ {}^{B}\dot{\mathbf{x}}_{P(o)}|_E\end{bmatrix}^{T}\cdot\begin{bmatrix}\mathbf{I}_{P(tra)} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{P(rot)}|_E\end{bmatrix}\cdot\begin{bmatrix}{}^{B}\dot{\mathbf{x}}_{P(pos)}|_B\\ {}^{B}\dot{\mathbf{x}}_{P(o)}|_E\end{bmatrix} \tag{26}$$

where,

$$\mathbf{I}_P|_E=\begin{bmatrix}\Im & \mathbf{0}\\ \mathbf{0} & \mathbf{J}_A\end{bmatrix}^{T}\cdot\begin{bmatrix}\mathbf{I}_{P(tra)} & \mathbf{0}\\ \mathbf{0} & \mathbf{I}_{P(rot)}|_B\end{bmatrix}\cdot\begin{bmatrix}\Im & \mathbf{0}\\ \mathbf{0} & \mathbf{J}_A\end{bmatrix} \tag{27}$$

is the total moving platform inertia matrix, expressed using the Euler angles system. In a compact form, results:

$$\mathbf{I}_P|_E=\mathbf{T}^{T}\cdot\mathbf{I}_P|_B\cdot\mathbf{T} \tag{28}$$

$$\mathbf{T}=\mathrm{diag}\left(\begin{bmatrix}\Im & \mathbf{J}_A\end{bmatrix}\right) \tag{29}$$

3.1.2 Actuators Kinetic Energy

As the actuators can only move perpendicularly to the base, their angular velocity relative to frame {B} is always zero. If the actuators are assumed to be equal, and the centre of mass of each actuator, cm_A, is located at a fixed distance, a_{cm}, from the actuator to fixed-length link connecting point (Figure 2), the position of the centre of mass relative to frame {B} is:

$${}^{B}\mathbf{p}_{A_i}|_B=\mathbf{b}_i+\left(l_i-a_{cm}\right)\cdot\mathbf{z}_B \tag{30}$$

where ${}^{B}\mathbf{p}_{A_i}|_B$ is a vector expressed in {B}.

The linear velocity of the actuator centre of mass, ${}^{B}\dot{\mathbf{p}}_{A_i}|_B$, relative to {B} and expressed in the same frame may be computed from the time derivative of the previous equation:

$${}^{B}\dot{\mathbf{p}}_{A_i}|_B=\dot{l}_i\cdot\mathbf{z}_B \tag{31}$$

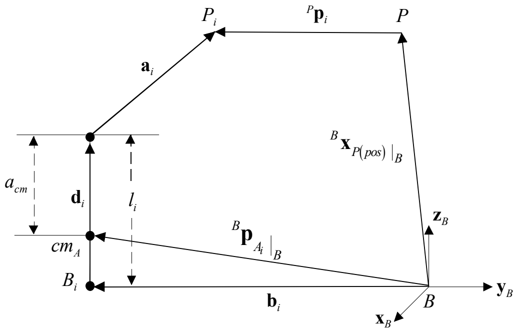

Fig. 2. Actuator centre of mass position.

The kinetic energy of each actuator is

$$K_{A_i} = \frac{1}{2}\cdot m_A \cdot {}^B\dot{\mathbf{p}}_{A_i}^T|_B \cdot {}^B\dot{\mathbf{p}}_{A_i}|_B = \frac{1}{2}\cdot m_A \cdot \dot{l}_i^2 \tag{32}$$

where m_A is the actuator mass.

Thus, using velocity kinematics:

$$\dot{\mathbf{l}} = \mathbf{J}_C \cdot \begin{bmatrix} {}^B\mathbf{v}_P|_B \\ {}^B\boldsymbol{\omega}_P|_B \end{bmatrix} = \mathbf{J}_C \cdot \mathbf{T} \cdot \begin{bmatrix} {}^B\dot{\mathbf{x}}_{P(pos)}|_B \\ {}^B\dot{\mathbf{x}}_{P(o)}|_E \end{bmatrix} \tag{33}$$

$$\dot{l}_i = \mathbf{J}_{F_i} \cdot \begin{bmatrix} {}^B\mathbf{v}_P|_B \\ {}^B\boldsymbol{\omega}_P|_B \end{bmatrix} = \mathbf{J}_{F_i} \cdot \mathbf{T} \cdot {}^B\dot{\mathbf{x}}_P|_{B|E} \tag{34}$$

$\mathbf{J}_{F_i}$ representing the jacobian, $\mathbf{J}_C$, i-line.

Equation (32) may be rewritten in the following form:

$$K_{A_i} = \frac{1}{2}\cdot m_A \cdot \begin{bmatrix} {}^B\mathbf{v}_P|_B \\ {}^B\boldsymbol{\omega}_P|_B \end{bmatrix}^T \cdot \mathbf{J}_{F_i}^T \cdot \mathbf{J}_{F_i} \cdot \begin{bmatrix} {}^B\mathbf{v}_P|_B \\ {}^B\boldsymbol{\omega}_P|_B \end{bmatrix} \tag{35}$$

where $\mathbf{I}_{A_i}|_{B(eq)} = m_A \mathbf{J}_{F_i}^T \mathbf{J}_{F_i}$ is a matrix expressed in {B}.

On the other hand, introducing matrix transformation $\mathbf{T}$ in equation (35) results in:

$$K_{A_i} = \frac{1}{2}\cdot m_A \cdot {}^B\dot{\mathbf{x}}_P^T|_{B|E} \cdot \mathbf{T}^T \cdot \mathbf{J}_{F_i}^T \cdot \mathbf{J}_{F_i} \cdot \mathbf{T} \cdot {}^B\dot{\mathbf{x}}_P|_{B|E} \tag{36}$$

$\mathbf{I}_{A_i}|_{E(eq)} = m_A \cdot \mathbf{T}^T \cdot \mathbf{J}_{F_i}^T \cdot \mathbf{J}_{F_i} \cdot \mathbf{T}$ being a matrix expressed in the Euler angles system.

It should be noted that matrices $\mathbf{I}_{A_i}|_{B(eq)}$ and $\mathbf{I}_{A_i}|_{E(eq)}$ represent the inertia matrices of a virtual moving platform that is equivalent, regarding its dynamic contribution, to each actuator.

Simultaneously considering the six actuators results in

$$K_A = \sum_{i=1}^{6} K_{A_i} = \frac{1}{2}\cdot m_A \cdot \sum_{i=1}^{6} \dot{l}_i^2 = \frac{1}{2}\cdot m_A \cdot \dot{\mathbf{l}}^T \cdot \dot{\mathbf{l}} \tag{37}$$

$$K_A = \frac{1}{2}\cdot m_A \cdot \begin{bmatrix} {}^B\mathbf{v}_P|_B \\ {}^B\boldsymbol{\omega}_P|_B \end{bmatrix}^T \cdot \mathbf{J}_C^T \cdot \mathbf{J}_C \cdot \begin{bmatrix} {}^B\mathbf{v}_P|_B \\ {}^B\boldsymbol{\omega}_P|_B \end{bmatrix} \tag{38}$$

$$K_A = \frac{1}{2}\cdot m_A \cdot {}^B\dot{\mathbf{x}}_P^T|_{B|E} \cdot \mathbf{T}^T \cdot \mathbf{J}_C^T \cdot \mathbf{J}_C \cdot \mathbf{T} \cdot {}^B\dot{\mathbf{x}}_P|_{B|E} \tag{39}$$

where

$$\mathbf{I}_A|_{B(eq)} = m_A \cdot \mathbf{J}_C{}^T \cdot \mathbf{J}_C \tag{40}$$

represents the inertia matrix of a virtual moving platform that is equivalent to the six actuators, expressed in the base frame. Expressing this matrix in the Euler angles system results in:

$$\mathbf{I}_A|_{E(eq)} = m_A \cdot \mathbf{T}^T \cdot \mathbf{J}_C{}^T \cdot \mathbf{J}_C \cdot \mathbf{T} \tag{41}$$

3.1.3 Fixed-length Links Kinetic Energy

Total kinetic energy of each fixed-length link, K_{L_i}, may be computed as the sum of two components: the translational kinetic energy, $K_{L_i(tra)}$ and the rotational kinetic energy, $K_{L_i(rot)}$.

If the centre of mass of each fixed-length link, cm_L, is located at a constant distance, b_{cm}, from the fixed-length link to moving platform connecting point (Figure 3), then its position relative to frame {B} is:

$$^B\mathbf{p}_{L_i}|_B = {}^B\mathbf{x}_{P(pos)}|_B + {}^P\mathbf{p}_i|_B - \frac{b_{cm}}{L}\cdot \mathbf{a}_i \tag{42}$$

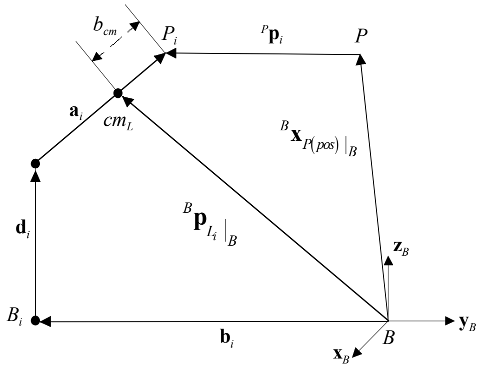

Fig. 3. Fixed-length link centre of mass position.

Equation (42) may be rewritten as:

$$\begin{aligned} ^B\mathbf{p}_{L_i}|_B &= \left(1-\frac{b_{cm}}{L}\right)\cdot {}^B\mathbf{x}_{P(pos)}|_B + \\ &\left(1-\frac{b_{cm}}{L}\right)\cdot {}^P\mathbf{p}_i|_B + \frac{b_{cm}}{L}\cdot \mathbf{b}_i + \frac{b_{cm}}{L}\cdot \mathbf{d}_i \end{aligned} \tag{43}$$

$^B\mathbf{p}_{L_i}|_B$ being a vector expressed in frame {B}.

The linear velocity of the fixed-length link centre of mass, $^B\dot{\mathbf{p}}_{L_i}|_B$, relative to {B} and expressed in the same frame may be computed from the time derivative of the previous equation:

$$^B\dot{\mathbf{p}}_{L_i}|_B = \left(1-\frac{b_{cm}}{L}\right)\cdot\left({}^B\dot{\mathbf{x}}_{P(pos)}|_B + {}^B\boldsymbol{\omega}_P|_B \times {}^P\mathbf{p}_i|_B\right) + \frac{b_{cm}}{L}\cdot \dot{l}_i \cdot \mathbf{z}_B \tag{44}$$

and, it can be rewritten as:

$$^B\dot{\mathbf{p}}_{L_i}|_B = \mathbf{J}_{B_i} \cdot \begin{bmatrix} {}^B\mathbf{v}_P|_B \\ {}^B\boldsymbol{\omega}_P|_B \end{bmatrix} \tag{45}$$

where the jacobian $\mathbf{J}_{B_i}$ is given by:

$$\mathbf{J}_{B_i} = \left(1-\frac{b_{cm}}{L}\right)\cdot \begin{bmatrix} 1 & 0 & 0 & 0 & {}^Pp_i|_{Bz} & -{}^Pp_i|_{By} \\ 0 & 1 & 0 & -{}^Pp_i|_{Bz} & 0 & {}^Pp_i|_{Bx} \\ \frac{b_{cm}}{L-b_{cm}}J_{Ci1} & \frac{b_{cm}}{L-b_{cm}}J_{Ci2} & \frac{b_{cm}}{L-b_{cm}}J_{Ci3}+1 & {}^Pp_i|_{By} + \frac{b_{cm}}{L-b_{cm}}J_{Ci4} & -{}^Pp_i|_{Bx} + \frac{b_{cm}}{L-b_{cm}}J_{Ci5} & \frac{b_{cm}}{L-b_{cm}}J_{Ci6} \end{bmatrix} \tag{46}$$

being $\mathbf{J}_{Cij}$ the elements of line i column j of matrix $\mathbf{J}_C$.

The translational kinetic energy of each fixed-length link is:

$$\begin{aligned} K_{L_i(tra)} &= \frac{1}{2}\cdot m_L \cdot {}^B\dot{\mathbf{p}}_{L_i}^T|_B \cdot {}^B\dot{\mathbf{p}}_{L_i}|_B \\ &= \frac{1}{2}\cdot m_L \cdot \begin{bmatrix} {}^B\mathbf{v}_P|_B \\ {}^B\boldsymbol{\omega}_P|_B \end{bmatrix}^T \cdot \mathbf{J}_{B_i}^T \cdot \mathbf{J}_{B_i} \cdot \begin{bmatrix} {}^B\mathbf{v}_P|_B \\ {}^B\boldsymbol{\omega}_P|_B \end{bmatrix} \end{aligned} \tag{47}$$

where m_L is its mass and

$$\mathbf{I}_{L_i(tra)}|_{B(eq)} = m_L \cdot \mathbf{J}_{B_i}^T \cdot \mathbf{J}_{B_i} \tag{48}$$

is a matrix expressed in the base frame.

Introducing transformation $\mathbf{T}$ in the previous equation results into:

$$K_{L_i(tra)} = \frac{1}{2} \cdot m_L \cdot {}^B\dot{\mathbf{x}}_P^T |_{B|E} \cdot \mathbf{T}^T \cdot \mathbf{J}_{B_i}^T \cdot \mathbf{J}_{B_i} \cdot \mathbf{T} \cdot {}^B\dot{\mathbf{x}}_P |_{B|E} \tag{49}$$

where

$$\mathbf{I}_{L_i(tra)} |_{E(eq)} = m_L \cdot \mathbf{T}^T \cdot \mathbf{J}_{B_i}^T \cdot \mathbf{J}_{B_i} \cdot \mathbf{T} \tag{50}$$

is a matrix expressed in the Euler angles system.

It should be noted that matrices $\mathbf{I}_{L_i(tra)} |_{B(eq)}$ and $\mathbf{I}_{L_i(tra)} |_{E(eq)}$ represent the inertia matrices of a virtual moving platform that is equivalent to each translational fixed-length link.

On the other hand, the rotational kinetic energy of each fixed-length link is:

$$K_{L_i(rot)} = \frac{1}{2} \cdot {}^B\boldsymbol{\omega}_{L_i}^T |_B \cdot \mathbf{I}_{L_i(rot)} |_B \cdot {}^B\boldsymbol{\omega}_{L_i} |_B \tag{51}$$

where ${}^B\boldsymbol{\omega}_{L_i} |_B$ represents the angular velocity of the fixed-length link, relative to {B} and expressed in the same frame, and $\mathbf{I}_{L_i(rot)} |_B$ represents the fixed-length link rotational inertia matrix, also expressed in frame {B}.

It is convenient to express the inertia matrix of the rotating fixed-length link in a frame fixed to the fixed-length link itself, $\{L_i\} \equiv \{\mathbf{x}_{L_i}, \mathbf{y}_{L_i}, \mathbf{z}_{L_i}\}$. So,

$$\mathbf{I}_{L_i(rot)} |_B = {}^B\mathbf{R}_{L_i} \cdot \mathbf{I}_{L_i(rot)} |_{L_i} \cdot {}^B\mathbf{R}_{L_i}^T \tag{52}$$

where ${}^B\mathbf{R}_{L_i}$ is the orientation matrix of each fixed-length link frame, $\{L_i\}$, relative to the base frame, {B}.

Fixed-length links frames were chosen in the following way: axis $\mathbf{x}_{L_i}$ coincides with the fixed-length link axis and points towards the fixed-length link to moving platform connecting point, meaning that it is coincident with vector $\mathbf{a}_i$; axis $\mathbf{y}_{L_i}$ is perpendicular to $\mathbf{x}_{L_i}$ and always parallel to the base plane (this condition being possible given the existence of a universal joint in the fixed-length link to actuator connecting point that negates any rotation along its own axis); axis $\mathbf{z}_{L_i}$ completes the frame according to the right hand rule, and its projection along axis $\mathbf{z}_B$ is always positive. Therefore, matrix ${}^B\mathbf{R}_{L_i}$ becomes:

$${}^B\mathbf{R}_{L_i} = \begin{bmatrix} \mathbf{x}_{L_i} & \mathbf{y}_{L_i} & \mathbf{z}_{L_i} \end{bmatrix} \tag{53}$$

where

$$\mathbf{x}_{L_i} = \begin{bmatrix} \frac{a_{ix}}{L} & \frac{a_{iy}}{L} & \frac{a_{iz}}{L} \end{bmatrix}^T \tag{54}$$

$$\mathbf{y}_{L_i} = \begin{bmatrix} -\frac{a_{iy}}{\sqrt{a_{ix}^2 + a_{iy}^2}} & \frac{a_{ix}}{\sqrt{a_{ix}^2 + a_{iy}^2}} & 0 \end{bmatrix}^T \tag{55}$$

$$\mathbf{z}_{L_i} = \mathbf{x}_{L_i} \times \mathbf{y}_{L_i} \tag{56}$$

So, the inertia matrices of the fixed-length links can be written as

$$\mathbf{I}_{L_i(rot)} |_{L_i} = \mathrm{diag}\left(\begin{bmatrix} I_{L_{xx}} & I_{L_{yy}} & I_{L_{zz}} \end{bmatrix}\right) \tag{57}$$

where $I_{L_{xx}}$, $I_{L_{yy}}$ and $I_{L_{zz}}$ are the fixed-length link moments of inertia, expressed in its own frame.

The angular velocity of each fixed-length link can be obtained from the linear velocities of two points. If these two points are taken as the fixed-length link to actuator, and the fixed-length link to moving platform connecting points, the following expression results:

$${}^B\boldsymbol{\omega}_{L_i} |_B \times \mathbf{a}_i = {}^B\mathbf{v}_P |_B + {}^B\boldsymbol{\omega}_P |_B \times {}^P\mathbf{p}_i |_B - \dot{l}_i \cdot \mathbf{z}_B \tag{58}$$

As the fixed-length link cannot rotate along its own axis the angular velocity along $\mathbf{x}_{L_i} = \hat{\mathbf{a}}_i$ is always zero and so vectors $\mathbf{a}_i$ and ${}^B\boldsymbol{\omega}_{L_i} |_B$ are always perpendicular. This enables equation (58) to be rewritten as:

$${}^B\boldsymbol{\omega}_{L_i} |_B = \frac{1}{L^2} \cdot \left[\mathbf{a}_i \times \left({}^B\mathbf{v}_P |_B + {}^B\boldsymbol{\omega}_P |_B \times {}^P\mathbf{p}_i |_B - \dot{l}_i \cdot \mathbf{z}_B\right)\right] \tag{59}$$

$${}^B\boldsymbol{\omega}_{L_i} |_B = \mathbf{J}_{D_i} \cdot \begin{bmatrix} {}^B\mathbf{v}_P |_B \\ {}^B\boldsymbol{\omega}_P |_B \end{bmatrix} \tag{60}$$

Therefore, the fixed-length link rotational kinetic energy will be:

$$K_{L_i(rot)} = \frac{1}{2} \cdot {}^B\dot{\mathbf{x}}_P^T |_B \cdot \mathbf{J}_{D_i}^T \cdot \mathbf{I}_{L_i(rot)} |_B \cdot \mathbf{J}_{D_i} \cdot {}^B\dot{\mathbf{x}}_P |_B \tag{61}$$

where,

$$\mathbf{I}_{L_i(rot)} |_{B(eq)} = \mathbf{J}_{D_i}^T \cdot \mathbf{I}_{L_i(rot)} |_B \cdot \mathbf{J}_{D_i} \tag{62}$$

is a matrix expressed in frame {B}.

Jacobian $\mathbf{J}_{D_i}$ is given by:

$$\mathbf{J}_{D_i} = \frac{1}{L^2} \cdot \begin{bmatrix} -a_{iy} J_{Ci1} & -a_{iy} J_{Ci2} - a_{iz} & a_{iy}(1 - J_{Ci3}) & a_{iy}\left({}^P p_i |_{By} - J_{Ci4}\right) + a_{iz} {}^P p_i |_{Bz} & -a_{iy}\left({}^P p_i |_{Bx} + J_{Ci5}\right) & -a_{iy} J_{Ci6} - a_{iz} {}^P p_i |_{Bx} \\ a_{iz} + a_{ix} J_{Ci1} & a_{ix} J_{Ci2} & -a_{iz}(1 - J_{Ci3}) & -a_{ix}\left({}^P p_i |_{By} - J_{Ci4}\right) & a_{iz} {}^P p_i |_{Bz} + a_{ix}\left({}^P p_i |_{Bx} + J_{Ci5}\right) & -a_{iz} {}^P p_i |_{By} + a_{ix} J_{Ci6} \\ -a_{iy} & a_{ix} & 0 & -a_{ix} {}^P p_i |_{Bz} & -a_{iy} {}^P p_i |_{Bz} & a_{ix} {}^P p_i |_{Bx} + a_{iy} {}^P p_i |_{By} \end{bmatrix} \tag{63}$$

Using transformation $\mathbf{T}$ in equation (61) results into:

$$K_{L_i(rot)} = \frac{1}{2} \cdot {}^B\dot{\mathbf{x}}_P^T |_{B|E} \cdot \mathbf{T}^T \cdot \mathbf{J}_{D_i}^T \cdot \mathbf{I}_{L_i(rot)} |_B \cdot \mathbf{J}_{D_i} \cdot \mathbf{T} \cdot {}^B\dot{\mathbf{x}}_P |_{B|E} \tag{64}$$

where

$$\mathbf{I}_{L_i(rot)} |_{E(eq)} = \mathbf{T}^T \cdot \mathbf{J}_{D_i}^T \cdot \mathbf{I}_{L_i(rot)} |_B \cdot \mathbf{J}_{D_i} \cdot \mathbf{T} \tag{65}$$

is a matrix expressed in the Euler angles system.

In a similar way as considered before, it should be noted that matrices $\mathbf{I}_{L_i(rot)} |_{B(eq)}$ and $\mathbf{I}_{L_i(rot)} |_{E(eq)}$ represent the inertia matrices of a virtual moving platform that is equivalent to each rotational fixed-length link.

3.2 Manipulator Potential Energy

The total potential energy, P, may be computed as the sum of the potential energies of all the rigid bodies comprising the mechanical system.

$$P = P_P + \sum_{i=1}^{6} P_{A_i} + \sum_{i=1}^{6} P_{L_i} \tag{66}$$

where P_P, P_{A_i} and P_{L_i} represent moving platform, actuator and fixed-length link potential energies, respectively.

Thus, using the base frame, {B}, as a datum, and assuming $\mathbf{z}_B \equiv -\hat{\mathbf{g}}$, the moving platform potential energy can be written as:

$$P_P = m_P \cdot g \cdot z_P \tag{67}$$

On the other hand, knowing the position of the centre of mass of each actuator relative to frame {B} (Figure 2), its potential energy, P_{A_i}, can be written as:

$$P_{A_i} = m_A \cdot g \cdot (l_i - a_{cm}) \tag{68}$$

Finally, using equation (43), the potential energy of each fixed-length link, will be:

$$P_{L_i} = m_L \cdot g \cdot \left[\left(1 - \frac{b_{cm}}{L}\right) \cdot z_P + \left(1 - \frac{b_{cm}}{L}\right) \cdot {}^P p_i |_{Bz} + \frac{b_{cm}}{L} \cdot l_i\right] \tag{69}$$

3.3 Manipulator Dynamic Equations

Using the manipulator kinetic and potential energy in the Lagrange equation, and separating the kinetic and gravitational force components, it would be possible to obtain the following matrix equations, with all matrices and vectors being expressed in the Euler angles system:

$$\mathbf{I}\big|_E\left({}^B\mathbf{x}_P\big|_{B|E}\right)\cdot{}^B\ddot{\mathbf{x}}_P\big|_{B|E}+\mathbf{V}\big|_E\left({}^B\mathbf{x}_P\big|_{B|E},{}^B\dot{\mathbf{x}}_P\big|_{B|E}\right)\cdot{}^B\dot{\mathbf{x}}_P\big|_{B|E}={}^P\mathbf{f}_{(kin)}\big|_{B|E} \quad (70)$$

$$\mathbf{G}\big|_E\left({}^B\mathbf{x}_P\big|_{B|E}\right)={}^P\mathbf{f}_{(gra)}\big|_{B|E} \quad (71)$$

$${}^P\mathbf{f}\big|_{B|E}={}^P\mathbf{f}_{(kin)}\big|_{B|E}+{}^P\mathbf{f}_{(gra)}\big|_{B|E} \quad (72)$$

$\mathbf{I}\big|_E$ represents system inertia matrix, $\mathbf{V}\big|_E$ represents system Coriolis and centripetal terms matrix, and $\mathbf{G}\big|_E$ represents system gravitational terms vector. Vector ${}^P\mathbf{f}_{(kin)}\big|_{B|E}$ represents the kinetic part of the generalized force acting in the moving platform, and ${}^P\mathbf{f}_{(gra)}\big|_{B|E}$ represents the gravitational component.

It should also be noted that:

$$\mathbf{I}\big|_E=\mathbf{I}_P\big|_E+\sum_{i=1}^{6}\mathbf{I}_{A_i}\big|_{E(eq)}+\sum_{i=1}^{6}\mathbf{I}_{L_i(tra)}\big|_{E(eq)}+\sum_{i=1}^{6}\mathbf{I}_{L_i(rot)}\big|_{E(eq)} \quad (73)$$

$$\mathbf{V}\big|_E=\mathbf{V}_P\big|_E+\sum_{i=1}^{6}\mathbf{V}_{A_i}\big|_{E(eq)}+\sum_{i=1}^{6}\mathbf{V}_{L_i(tra)}\big|_{E(eq)}+\sum_{i=1}^{6}\mathbf{V}_{L_i(rot)}\big|_{E(eq)} \quad (74)$$

$$\mathbf{G}\big|_E=\mathbf{G}_P\big|_E+\sum_{i=1}^{6}\mathbf{G}_{A_i}\big|_E+\sum_{i=1}^{6}\mathbf{G}_{L_i}\big|_E \quad (75)$$

Rigid body inertia matrices may be directly obtained from the kinetic energy expressions, computed previously. Coriolis and centripetal terms matrices can be computed throughout the known inertia matrices. Finally, gravitational force component may be obtained using system potential energy.

Generally speaking, if $\mathbf{I}(\mathbf{x})$ is an inertia matrix, then, it is known, the respective Coriolis and centripetal terms matrix, $\mathbf{V}(\mathbf{x},\dot{\mathbf{x}})$, may be computed as [11]:

$$\mathbf{V}(\mathbf{x},\dot{\mathbf{x}})=\frac{1}{2}\cdot\left[\dot{\mathbf{I}}(\mathbf{x})+\mathbf{U}^T-\mathbf{U}\right] \quad (76)$$

where $\mathbf{U}$ is given by

$$\mathbf{U}=\left(\mathfrak{I}\otimes\dot{\mathbf{x}}^T\right)\cdot\frac{\partial\mathbf{I}(\mathbf{x})}{\partial\mathbf{x}} \quad (77)$$

and

$$\frac{\partial\mathbf{I}(\mathbf{x})}{\partial\mathbf{x}}=\begin{bmatrix}\frac{\partial\mathbf{I}}{\partial x} & \frac{\partial\mathbf{I}}{\partial y} & \cdots & \frac{\partial\mathbf{I}}{\partial\varphi}\end{bmatrix}^T \quad (78)$$

$$\left(\mathfrak{I}\otimes\dot{\mathbf{x}}^T\right)=\begin{bmatrix}\dot{\mathbf{x}}^T & & (0)\\ & \dot{\mathbf{x}}^T & \\ (0) & & \dot{\mathbf{x}}^T\end{bmatrix} \quad (79)$$

Matrix $\left(\mathfrak{I}\otimes\dot{\mathbf{x}}^T\right)$ is 6×36, and $\partial\mathbf{I}(\mathbf{x})/\partial\mathbf{x}$ is 36×6. Symbol $\otimes$ represents Kronecker product.

3.4 Simplified Dynamic Model

Computation of the fixed-length links dynamic contribution requires a high computational effort. Thus, a simplification to the presented complete dynamic model is now proposed.

Each fixed-length link is modeled as a zero-mass virtual link connecting two point masses located at its ends. This is a reasonable simplification as the fixed-length links are connected to the moving platform and to the actuators by steel universal joints.

So, the fixed-length link kinetic energy will be:

$$K_{L_i}=K_{L_i(tra)}+K_{L_i(rot)} \quad (80)$$

Preserving the centre of mass location, results in:

$$K_{L_i(tra)}=\frac{1}{2}\cdot\left(m_{L1}+m_{L2}\right)\cdot{}^B\dot{\mathbf{p}}_{L_i}^T\big|_B\cdot{}^B\dot{\mathbf{p}}_{L_i}\big|_B \quad (81)$$

where m_{L1} is a point mass located at the connection point between the moving platform and the fixed-length link, and m_{L2} is a point mass located at the connection point between the actuator and the fixed-length link: $m_L=m_{L1}+m_{L2}$.

Position and velocity of the centre of mass expressed in {B} are,

$${}^B\mathbf{p}_{L_i}\big|_B=\frac{m_{L1}\cdot\left({}^B\mathbf{x}_{P(pos)}\big|_B+{}^P\mathbf{p}_i\big|_B\right)+m_{L2}\cdot\left(\mathbf{b}_i+\mathbf{d}_i\right)}{m_{L1}+m_{L2}} \quad (82)$$

$${}^B\dot{\mathbf{p}}_{L_i}\big|_B=\frac{m_{L1}\cdot\left({}^B\mathbf{v}_P\big|_B+{}^B\boldsymbol{\omega}_P\big|_B\times{}^P\mathbf{p}_i\big|_B\right)+m_{L2}\cdot\dot{\mathbf{d}}_i}{m_{L1}+m_{L2}} \quad (83)$$

Rotational kinetic energy will be:

$$K_{L_i(rot)}=\frac{1}{2}\cdot{}^B\dot{\mathbf{x}}_P^T\big|_{B|E}\cdot\mathbf{T}^T\cdot\mathbf{J}_{D_i}^T\cdot{}^B\mathbf{R}_{L_i}\cdot\mathbf{I}_{L_i(rot)}\big|_{L_i}\cdot{}^B\mathbf{R}_{L_i}^T\cdot\mathbf{J}_{D_i}\cdot\mathbf{T}\cdot{}^B\dot{\mathbf{x}}_P\big|_{B|E} \quad (84)$$

where

$$\mathbf{I}_{L_i(rot)}\big|_{L_i}=\begin{bmatrix}0 & 0 & 0\\ 0 & a^2m_{L2}+b^2m_{L1} & 0\\ 0 & 0 & a^2m_{L2}+b^2m_{L1}\end{bmatrix} \quad (85)$$

and $a=L-b$; $b=L\cdot m_{L2}\cdot\left(m_{L1}+m_{L2}\right)^{-1}$.

It should be noted this is equivalent to consider m_{L1} and m_{L2} as part of the moving platform and actuators, respectively. Therefore, the fixed-length links need not be modeled as independent rigid bodies, being their masses distributed between the moving platform and actuators:

$$m_{A(new)}=m_A+m_{L2} \quad (86)$$

$$m_{P(new)}=m_P+6m_{L1} \quad (87)$$

$$\mathbf{I}_{P(rot)(new)}\big|_P=\mathbf{I}_{P(rot)}\big|_P+\mathrm{diag}\left(\left[I_{xx}\quad I_{yy}\quad I_{zz}\right]\right) \quad (88)$$

$$I_{xx}=2m_{L1}r_P^2\left[\sin^2\left(\frac{\pi}{3}-\phi_P\right)+\sin^2\left(\frac{\pi}{3}+\phi_P\right)+\sin^2\left(\phi_P\right)\right] \quad (89)$$

$$I_{yy}=2m_{L1}r_P^2\left[\cos^2\left(\frac{\pi}{3}-\phi_P\right)+\cos^2\left(\frac{\pi}{3}+\phi_P\right)+\cos^2\left(\phi_P\right)\right] \quad (90)$$

$$I_{zz}=6m_{L1}r_P^2 \quad (91)$$

Table 1 presents the computational effort involved in the computation of the complete dynamic model. That is, the number of arithmetic operations involved in the computation of the Inertia and Coriolis and centripetal terms matrices (gravitational terms are not presented, as they present negligible computational burden).

It should be noted the simplified model is computationally much more efficient, as the computation of the matrices requiring the largest relative computational effort: the Inertia and Coriolis and centripetal terms matrices associated with the fixed-length links are not needed.

Table 1. Number of arithmetic operations involved in the computation of the Inertia and Coriolis and centripetal terms matrices.

	Coriolis and centripetal matrices terms			Inertia matrices terms		
	Add.	Mult.	Div.	Add.	Mult.	Div.
Mobile platform	310	590	0	36	90	0
Six actuators	3028	4403	30	297	384	12
Translating links	**4506**	**9474**	**36**	**378**	**1038**	**30**
Rotating links	**13080**	**22266**	**42**	**900**	**1758**	**18**
Total operations	20924	36733	108	1611	3270	60

4 Numerical Example

A 6-dof parallel manipulator presenting the kinematic and dynamic parameters shown in Table 2 was considered.

Table 2. Manipulator parameters.

Para.	Value	Para.	Value	Para.	Value
r_B	1.500 m	m_P	1.430 kg	I_{Lxx}	0.0 kg·m^2
r_P	0.750 m	m_A	0.123 kg	I_{Lyy}	0.1 kg·m^2
L	1.837 m	m_L	0.389 kg	I_{Lzz}	0.1 kg·m^2
ϕ_B	15º	I_{Pxx}	0.2 kg·m^2	m_{L1}	0.194 kg
ϕ_P	0º	I_{Pyy}	0.2 kg·m^2	m_{L2}	0.194 kg
		I_{Pzz}	0.4 kg·m^2	b_{cm}	0.918 m

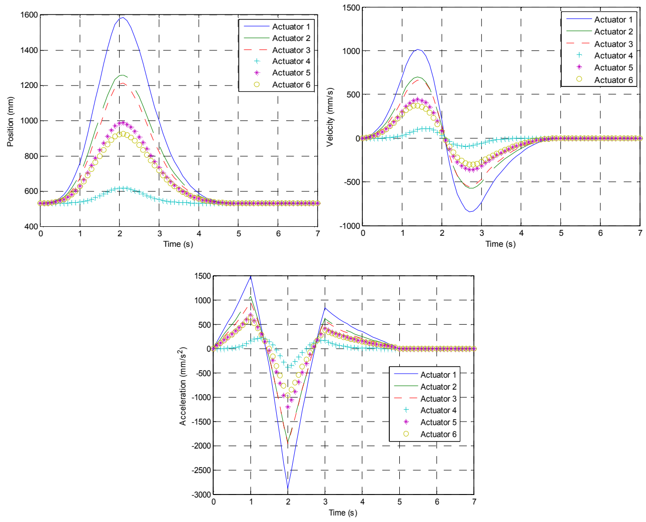

Fig. 4. Actuators trajectories.

A trajectory was specified in task space. The moving platform initial position is $P_1 = [0, 0, 2000, 0, 0, 0]$ (mm; deg). The moving platform is then displaced to point $P_2 = [-100, -200, 2500, 15, -15, 15]$ (mm; deg), and finally it returns to point P_1.

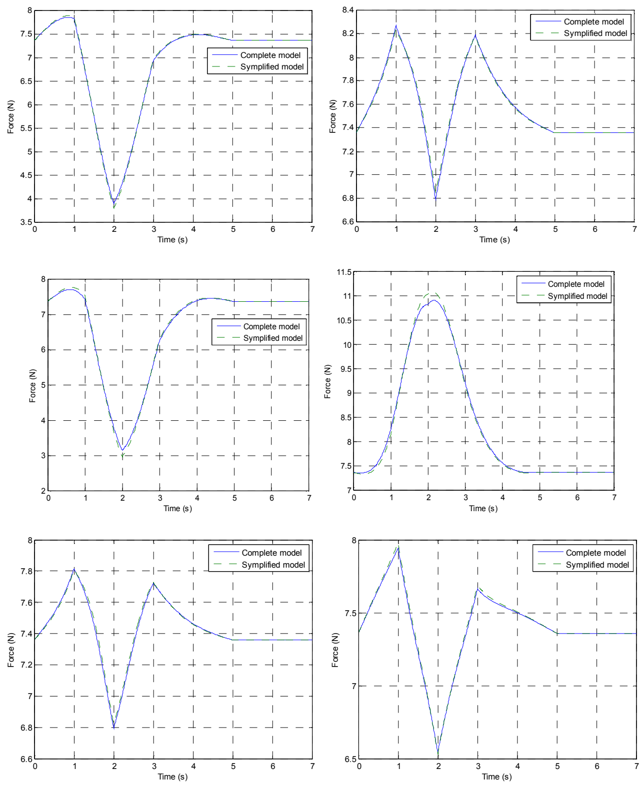

Fig. 5. Actuators forces: complete *vs* simplified models.

Third order trigonometric splines were interpolated between the specified points, in order to obtain smooth and continuous trajectories. Figure 4 shows the corresponding actuators trajectories.

Figure 5 shows the actuators developed forces necessary to follow the specified trajectories. It should be noted the computed actuators forces are similar for both the complete and the simplified dynamic models, illustrating the suitability of the proposed simplified model.

5 Conclusion

Dynamic modeling of parallel manipulators presents an inherent complexity. Despite the intensive study in this topic of robotics, mostly conducted in the last two decades, additional research still has to be done.

In this paper the complete dynamic model of a 6-dof parallel manipulator was presented, based on the Lagrange's formulation. The approach is completely general, and can be used as a dynamic modelling tool applicable to any mechanism.

The involved computational effort was evaluated and compared with the one presented by a simplified model. The proposed simplified model presents a much lower computational burden, being representative of the mechanical behavior of the manipulator.

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