NECESSARY CONDITIONS OF OPTIMALITY FOR CONSTRAINED INFINITE HORIZON DIFFERENTIAL INCLUSIONS

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Abstract

This article presents and discusses necessary conditions of optimality for infinite horizon dynamic optimization problems whose cost functional depends on the state variable at the final time and the dynamics are given by a differential inclusion. Moreover, the state variable satisfies constraints at both endpoints and the optimization is carried out over asymptotically convergent state trajectories. The novelty of the proposed optimality conditions for this class of problems is that the boundary condition of the adjoint variable is given as a weak directional inclusion at infinity. This allows to overcome some of the drawbacks of existing necessary conditions of optimality for infinite horizon problems.

Key words

Infinite horizon dynamic optimization, Necessary conditions of Optimality, Differential inclusions, transversality conditions.

1 Introduction

This article concerns necessary conditions of optimality for infinite horizon dynamic optimization problems. This problem has been considered since the early seventies and a lot of effort has been spent on how to define the transversality condition to be satisfied by the adjoint variable so that the conditions remain informative.

The challenges posed by transversality conditions in infinite horizon control problems were already identified in [Halkin, 1974] where a problem with an integral cost functional was considered. After defining an appropriate solution concept, a maximum principle without transversality conditions is derived. Later, it is shown in [Michel, 1982], that, under a certain controllability assumption, the Hamiltonian tends to zero as time goes to infinity. Inspired by stability theory, a regularity assumption formulated in terms of Lyapunov exponents to be satisfied by the adjoint variable is required in [Smirnov, 1996] in order to derive necessary and sufficient conditions of optimality for infinite horizon control problems with a transversality condition. A nonsmooth maximum principle encompassing final time transversality conditions was derived in [Seierstad, 1999] for nonsmooth optimal control problems with final state dependent cost functional as well as final time state constraints. However, a linear structure is required for both of these ingredients. In [Weber, 2006], strong hypotheses implying that the adjoint variable remains bounded were assumed on the data of an infinite horizon discounted optimal control problem in order to derive a maximum principle with a transversality condition.

These results reveal the essential intrinsic challenges of deriving necessary conditions of optimality with transversality conditions for infinite horizon optimal control problems with either final state constraints or final state dependent cost functional: in order to propagate in a informative way the final time boundary condition of the adjoint variable to any given finite time, one needs to impose very strict assumptions and this implies restricting the range of applicability of the derived optimality conditions.

Following the work in [Pereira and Silva, 2006], the aim of this work is to show that a new weaker notion of transversality condition, first introduced in [Pereira and Silva, 2011], allows to improve the necessary conditions of optimality for dynamic optimization problems with dynamics given by a differential inclusions and constraints on the state trajectory endpoints with assumptions that are usually considered in the literature for finite horizon problems.

We consider the following optimization problem over asymptotically convergent state trajectories:

$$(P)\ \text{Minimize}\quad g(\xi) \tag{1}$$

$$\text{subject to}\quad \dot{x}(t) \in F(t, x(t)),\ \mathcal{L} - a.e. \tag{2}$$

$$(x(0), \xi) \in C_0 \times C_\infty, \tag{3}$$

$$\xi = \lim_{t\to\infty} x(t), \tag{4}$$

where $g : \boldsymbol{R}^n \to \boldsymbol{R}$, $F : [0,\infty) \times \boldsymbol{R}^n \to \mathcal{P}(\boldsymbol{R}^n)$, C_0 and C_∞ are compact sets, and F is compact and convex valued set-valued map.

The article is organized as follows. In the next section we refine some preliminary concepts first introduced in [Pereira and Silva, 2011]. These encompass the basis on which our notion of transversality condition builds as well as the class of solutions on which the optimization is performed. Then, in section 3, we present and discuss the optimality conditions as well as the assumptions required for the data of the problem. An outline of the proof is given in section 4 and the conclusions presented in the last section.

2 Preliminary definitions

Since the optimization is carried out over all feasible control processes that converge asymptotically to some point in the infinite time horizon, we need to define equilibrium at infinity. We say that the point $\xi \in \boldsymbol{R}^n$ is an equilibrium as $t \to \infty$ if $\exists$ a feasible trajectory $(x(\cdot)$ such that

$$\lim_{t\to\infty} x(t) = \xi, \text{ and } 0 \in \liminf_{t\to\infty} F(t, x(t)),$$

where the this limit set is assumed to be nonempty.

In order to capture the behavior of the adjoint variable as time goes to infinity, we specify the final endpoint transversality condition in terms of directional inclusion at infinity. This will call for a number of concepts introduced in [Rockafellar, Wets, 1997] enabling to deal with the extended $\boldsymbol{R}^n$.

We consider a direction to be a ray, i.e., a closed half-line emanating from the origin. We think of rays as abstract direction points which lie beyond $\boldsymbol{R}^n$ and form the horizon of $\boldsymbol{R}^n$, denoted by hzn $\boldsymbol{R}^n$. We represent a direction point by dir$x$, where x is any nonzero vector in the ray representing the direction point in question. The cosmic space csm $\boldsymbol{R}^n$ is the union of the $\boldsymbol{R}^n$ with its horizon hzn$\boldsymbol{R}^n$. With this definition it becomes clear that the cosmic $\boldsymbol{R}^n$ is a compact space.

A sequence of points $x_k \in \boldsymbol{R}^n$ converges to a direction point dir x, written $x_k \to$ dir x, $x \neq 0$, if $\lambda_k x_k \to x$ for some choice of $\lambda_k \searrow 0$, i.e., $\lambda_k > 0$ and $\lambda_k \to 0$.

Given a set $C \subset \boldsymbol{R}^n$ the cosmic closure closure of C is given by

$$\text{csm}C := \text{cl}C \cup \text{hzn}C,$$

where cl$C$ is the usual closure of C in $\boldsymbol{R}^n$ while the hzn$C$ is the collection of all direction points obtained with limits of sequences of points in C.

Given a cone $K \subset \boldsymbol{R}^n$, denote the set of of direction points represented by the rays of K by dir$K$.

For a given nonempty set C in $\boldsymbol{R}^n$, the horizon cone representing the direction set hzn$C$, is defined by

$$C^\infty = \{x : \exists x_k \in C, \lambda_k \searrow 0, \text{ with } \lambda_k x_k \to x\}.$$

Observe that C is bounded if and only if $C^\infty = \{0\}$. With this notation, we have that hzn$C$ = dirC^∞ and csm$C$ = cl$C \cup$ dirC^∞.

A subset of csm $\boldsymbol{R}^n$, written as $C \cup$ dir$K$, for a set $C \subset \boldsymbol{R}^n$ and a cone $K \subset \boldsymbol{R}^n$, is closed in csm $\boldsymbol{R}^n$ if C and K are closed in $\boldsymbol{R}^n$ and $C^\infty \subset K$. The cosmic closure of $C \cup$ dir$K$ is

$$\text{csm}(C \cup \text{dir}K) = \text{cl}C \cup \text{dir}(C^\infty \cup \text{cl}K).$$

Now, we are in position to define the concept of directional inclusion at infinity. This enables us to state boundary conditions involving variables which may either become unbounded or persist in a certain set as time goes to infinity.

Let $y : [0,\infty) \to \boldsymbol{R}^n$ be a continuous function. Let $\boldsymbol{P}(y) := \boldsymbol{P}_L(y) \cup \text{dir}\boldsymbol{P}_\infty(y)$, also alluded to as the set of persistency points of y, where

$$\boldsymbol{P}_L(y) := \{\xi \in \boldsymbol{R}^n : \exists\, t_i \to \infty,\ \lim_{i\to\infty} y(t_i) = \xi\}$$
$$\text{dir}\boldsymbol{P}_\infty(y) := \{\xi \in \boldsymbol{R}^n : \exists t_i \to \infty, \lambda_i \searrow 0, \lim_{i\to\infty} \lambda_i y(t_i) = \xi\}.$$

Given a function $y : [0,\infty) \to \boldsymbol{R}^n$ and a set $C \subset \boldsymbol{R}^n$ we say that y satisfies the weak directional inclusion in C at ∞ if $\boldsymbol{P}(y) \cap \text{csm}\, C \neq \emptyset$. This relation can be referred to in short notation by $y \in^*_\infty C$.

3 Necessary conditions of optimality

This problem is cast in the context of nonsmooth analysis (see [Clarke, Ledyaev, Stern, Wolenski, 1998]) due to both the assumptions on its data and the approach used to derive the optimality conditions.

We consider the following assumptions on the data of the problem:

H1 g is continuously differentiable.

H2 F measurably Lipschitz, i.e., F is Lebesgue measurable with respect to time and Lipschitz continuous in x, $\forall\, t \in [0,\infty)$.

H3 The endpoint state constraint sets C_0, and C_∞ are closed.

H4 The $\lim_{t\to\infty} F(t, x(t))$ exists in the sense of Hausdorff and is denoted by $F_\infty(\xi)$ where $\xi := \lim_{t\to\infty} x(t)$.

H5 Let $\xi^* := \lim_{t\to\infty} x^*(t)$. There exists $\delta > 0$ such that $\forall\, x \in \xi^* + \delta B$,

$$0 \in \text{Int} \lim_{t\to\infty} F(t, x).$$

H6 $\exists\, v_0 \in \boldsymbol{R}^n$ such that $v_0 \in \text{Int}F(0, x^*(0))$ and

$$\begin{cases} \text{either } x^*(0) \in \text{Int}C_0, \\ \text{or } \langle \zeta_0, v_0 \rangle < 0,\ \forall \zeta_0 \in N_{C_0}(x^*(0)). \end{cases}$$

Although assumption $(H1)$ can be weakened to mere Lipschitz continuity, we keep it in order to facilitate some developments discussed later in this article. With the $(H1)$ weakened to Lipschitz continuity, $(H1)$ –

$(H3)$ are the standing assumptions usually considered for finite horizon problems. $(H4)-(H6)$ are additional technical assumptions required o prove the stated necessary conditions of optimality. $(H4)$ reflects a sort of persistence of the velocity set at the limiting value of the state variable, $(H5)$ implies the controllability in a neighborhood of the optimal reference trajectory as time goes to ∞ and $(H6)$ is an initial point controllability condition with respect to the initial state constraint.

Our necessary conditions of optimality for (P) are stated in the form of a maximum principle and they involve the Hamiltonian defined as

$$H(t, x, p) := \sup\{\langle p, v\rangle : v \in F(t, x)\}.$$

The adjoint variable $p : [0, \infty) \to \boldsymbol{R}^n$ satisfies a boundary condition at $t = \infty$. This is stated as the existence of a non empty subset of its persistency points, $\boldsymbol{P}(p)$, on the cosmic closure of the right hand set of the usual transversality condition. Moreover p can be regarded as a subgradient of the value function V, defined by

$$V(t, z) := \text{Min}\{g(\xi) : \text{ over all feasible } x \text{ s. t. } \lim_{\tau\to\infty} x(\tau) = \xi,\ x(t) = z\},$$

along the optimal trajectory. In particular, if p converges asymptotically to some point $\bar{p}$, then $\boldsymbol{P}(p) = \{\bar{p}\}$. If p approaches a limit cycle C_L at infinite time, then $\boldsymbol{P}(p) = C_L$. The pattern of realization of the limiting approach towards a given infinitely often visited set of points might not be periodic.

Next, we state the main result of this article.

Theorem. Let x^* be a solution to (P). Then, there exists a multiplier (p, λ_0), with $\lambda_0 \geq 0$, satisfying:

a) $\lambda_0 + \|p\| \neq 0$ (nontriviality).
b) $\exists p(0) \in N_{C_0}(x^*(0))$ for which there is a solution to
$$-\dot{p}(t) \in \partial_x H(t, x^*(t), p(t)),\ \mathcal{L}\text{ -a.e.},$$
satisfying:
$$(i) - p(t) \in \partial_x V(t, x^*(t)),\ \mathcal{L}\text{-a.e. on } [0, \infty);$$
and $(ii)\boldsymbol{P}(-p) \cap \text{csm}(\lambda_0\partial g(\xi^*) + N_{C_\infty}(\xi^*)) \neq \emptyset$.

Remark that $\boldsymbol{P}(-p) \cap \text{csm}\ (\lambda_0\partial g(\xi^*) + N_{C_\infty}(\xi^*))$ can be interpreted as $\exists \zeta \in \lambda_0\partial g(\xi^*) + N_{C_\infty}(\xi^*)$ for which

either $\zeta \in \boldsymbol{P}_L(-p)$, if p is bounded,
or $\zeta \in \text{dir}\boldsymbol{P}^\infty(p)$, otherwise.

The information provided by this concept is certainly weaker than that given by the boundary condition of the adjoint variable in the finite time interval context. In general there are many functions p that persist in an absolute or a directional sense towards a point of $\lambda_0\partial g(\xi^*) + N_{C_\infty}(\xi^*)$ at infinite time. Nevertheless, this information is still useful in delimiting the number of multipliers which satisfy the maximum condition.

4 Outline of the proof

The proof is based on extracting the limit of a subsequence of multipliers associated with the corresponding sequence of solutions to a family of auxiliary finite horizon optimal control problems converging to (P).

Take $\{T_k\}$, $T_k \uparrow \infty$ and consider the following auxiliary problem.

$$(P_{T_k})\ \text{Minimize}\ V(T_k, x(T_k))$$
$$\text{subject to}\ \dot{x} \in F(\tau, x),\ \mathcal{L}\text{-a.e. in } [0, T_k],$$
$$x(0) \in C_0,$$

where $V(t, z) : [0, \infty) \times \boldsymbol{R}^n \to \boldsymbol{R}$ is defined by

$$V(t, z) := \min\{g(\xi) : \dot{x} \in F(\tau, x),\ \mathcal{L}\text{-a.e. in } [t, \infty),\ x(t) = z,\ \lim_{\tau\to\infty} x(\tau) = \xi \in C_\infty\}.$$

Notice that, by the principle of optimality, the solution to (P_{T_k}), denoted by x_k^* coincides with x^* on $[0, T_k]$.

Before pursuing, we would like to point out the following properties of V (which follow from [Vinter, 2000]).

Proposition 2.

(i) V is measurable in t and Lipschitz continuous on z for any finite time.
(ii) $\liminf_{t\to\infty} V(t, z) = \begin{cases} g(z) & \text{if } z \in C_\infty \\ +\infty & \text{otherwise} \end{cases}$
(iii) $\lim_{t\to\infty} V(t, x^*(t)) = g(\xi^*)$ uniformly.

The various items of this proposition are straightforward extensions, under the hypotheses assumed in this article, of the corresponding finite horizon counterparts. It should be remarked that item (i) draws heavily from the controllability at infinity in order to ensure the Lipschitz dependence instead of the usual lower semicontinuity usually proved for finite horizon problems whose state variable is constrained at the final time.

From the principle of optimality we can see that the sequence of trajectories $x_{T_k}^*$, each one associated with the solution to (P_{T_k}), converges uniformly to the optimal trajectory to the original infinite horizon problem (P). In this sense, we can say that the sequence of finite horizon problems (P_{T_k}) approximates (P).

Next, we apply the the Maximum Principle to (P_{T_k}), and we obtain a multiplier p_k satisfying

$$-\dot{p}_k(t) \in \partial_x H_k(t, x^*(t), p_k(t)),\ [0, T_k]\text{-a.e.}$$
$$p_k(0) \in N_{C_0}(x^*(0)),$$
$$-p_k(T_k) \in \partial_x V(T_k, x^*(T_k)),$$
$$\dot{x}^*(t) \text{ maximizes in } F(t, x^*(t)) \text{ the map}$$
$$v \to \langle p_k(t), v\rangle,\ [0, T_k]\text{-a.e..}$$

Now, let us compute an estimate of $\partial_x V(T_k, x^*(T_k))$.

Proposition 3. Under the assumptions $(H1) - (H6)$, we have that $\partial_x V(T_k, x^*(T_k))$ contains the set

$$\{\bar{p}_k \in \boldsymbol{R}^n : \exists(\bar{p}, \bar{\lambda}) \text{ satisfying :}$$
$$(i)\ \|\bar{p}(\cdot)\| + \bar{\lambda} \neq 0,\ \bar{\lambda} \geq 0$$
$$(ii) - \dot{\bar{p}}(t) \in \partial_x H(x^*(t), \bar{p}(t)),\ [T_k, \infty)\text{-a.e.}$$
$$(iii)\ \bar{p}(T_k) = \bar{p}_k$$
$$(iv)\ \boldsymbol{P}(-\bar{p}) \cap \text{csm}\,(\lambda_0 \partial g(\xi^*) + N_{C_\infty}(\xi^*)) \neq \emptyset$$
$$(v)\ \dot{x}^*(t) \text{ maximizes in } F(t, x^*(t)),\ [T_k, \infty)\text{-a.e., the map } v \to \langle \bar{p}(t), v\rangle\}$$

Let $x \in AC([0, \infty); \boldsymbol{R}^n)$ be such that $x(T_k) = z$, $\dot{x}(t) \in F(t, x(t))$ a.e. and $\lim_{t\to\infty} x(t) = \xi$ asymptotically. Then, by using the fact that h is assumed to be C_1, we have

$$g(\xi) = g(z) + \int_{T_k}^{\infty} \nabla g(x(t))\dot{x}(t)dt.$$

We also need an additional auxiliary variable y satisfying $\dot{y} = 0$ with $y(T_k) \in C_\infty$ and also $\lim_{t\to\infty}(y(t) - x(t)) = 0$. Note that, since $\tilde{C} := \{(x, y) : x = y\}$, we have that, for any $(x, y) \in \tilde{C}$, $N_{\tilde{C}}(x, y) = \{(\bar{p}_x, \bar{p}_y) : \bar{p}_x = -\bar{p}_y\}$.

Now, notice that $V(T_k, z)$ is the minimum cost of the following auxiliary optimal control problem

$$\begin{aligned} &\text{Minimize } \int_{T_k}^{\infty} \nabla g(x(t))\dot{x}(t)dt \\ &\text{subject to } \dot{x}(t) \in F(t, x(t)),\ \dot{y}(t) = 0,\ [T_k, \infty)\text{-a.e.} \\ &\qquad \lim_{t\to\infty}(x(t), y(t)) \in \tilde{C} \\ &\qquad (x(T_k), y(T_k)) \in \{z\} \times C_\infty \end{aligned}$$

Observe that the generalized gradient of V with respect to x at time T_k at $x^*(T_k)$ is given by the set of values of the (symmetric of the) adjoint variable at time T_k. Remark also that the cost functional of this problem does not depend on state at infinite time. The final endpoint constraint does not cause any difficulty since it is affine in the state variable and always active.

By applying the maximum principle to this auxiliary problem, and, then, by expressing the obtained conditions in terms of the data of the original problem, it is straightforward to derive the intended characterization of the estimate of $\partial_x V(T_k, x^*(T_k))$.

Indeed, we have

$$H(t, x, y, p_x, p_y, \lambda_0) = \sup_{v\in F(t,x)} \{\langle p_x, v\rangle - \lambda_0 \nabla g(x)v\}$$

and, thus:

$$-\dot{p}_x(t) \in \partial_x H(t, x^*(t), y^*(t), p_x(t), p_y(t), \lambda_0).$$
$$-\dot{p}_y(t) \equiv 0, \text{ and } p_y(t) \equiv p_y(T_k) \in N_{C_\infty}(x^*(T_k)).$$

$\exists\{t_i\}$, $t_i \uparrow \infty$, $\exists\{\alpha_i\}$, $\alpha_i > 0$, $\alpha_i \to \alpha_\infty \geq 0$, such that

$$\lim_{i\to\infty} \alpha_i p_x(t_i) = -p_y(T_k).$$

Notice that the third item arises naturally from the fact that the adjoint equation relative to p_x, involving also λ_0, can be scaled down by some positive number.

Now, by putting $p(t) = p_x(t) - \lambda_0 \nabla g(x^*(t))$, we have that

$$-\dot{p}(t) \in \partial_x H(t, x^*(t), p(t)),$$

and, by considering sequences $\{t_i\}$ and $\{\alpha_i\}$ with either $\alpha_\infty > 0$ or $\alpha_\infty = 0$ we have the stated transversality condition.

To complete the proof of Theorem 1, it is enough to show that the desired conditions are obtained as the limit of the necessary conditions derived for (P_{T_k}). By once more recalling the principle of optimality, the properties of V, and using the characterization of the estimate of its generalized gradient derived in the above proposition, we readily obtain the desired conclusions, i.e., the necessary conditions of optimality for (P).

5 Conclusions

In this article, necessary conditions of optimality in the form of the Hamiltonian inclusions and featuring a novel transversality condition were given for an infinite horizon dynamic optimization problem with dynamics given by a differential inclusion and whose state trajectories are required to converge asymptotically to an equilibrium point is constrained to a given closed set. Various comments relating the obtained result are included.

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