Optimal unambiguous discrimination problems under different a priori knowledge

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We have recently explored optimum unambiguous discrimination problems under different *a priori* knowledge. In general, *a priori* knowledge in optimum unambiguous discrimination problems can be classed into two types: *a priori* knowledge of discriminated states themselves and *a priori* probabilities of preparing the states. It is further clarified that no matter whether a priori probabilities of preparing discriminated states are available or not, what type of discriminators one should design just depends on what kind of the knowledge of discriminated states. This is in contrast to the observation that choosing the parameters of discriminators relies on both types of *a priori* knowledge.

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I. INTRODUCTION

Recently, quantum information and quantum computation is the focus of research, and a great progress of quantum information has been made in both theoretical and experimental aspects[1]. However, a careful thinker may critically ask if there is any special problems in the domain of quantum information processing from the view point of decision theory. Recently, we attempted to give an interesting answer: *A priori* knowledge plays special role in the domain of quantum information processing. Since quantum states discrimination[2–5] is very fundamental in the domain of quantum information processing[6], it is reasonable to carefully explore optimum unambiguous discrimination problems under various *a priori* knowledge. To authors' knowledge, optimum unambiguous discrimination problems have not been thoroughly investigated under various *a priori* knowledge so far. The rest of this paper is organized as follows. In Sect. II, we review the results on optimal unambiguous discrimination with the knowledge of the preparation probabilities of two discriminated states. Furthermore, we study the optimal unambiguous discrimination without *a priori* probabilities of preparation two discriminate states in Sect. III. The paper briefly concludes with Sect. IV.

II. OPTIMAL UNAMBIGUOUS DISCRIMINATION PROBLEMS WITH KNOWLEDGE OF *A PRIORI* PREPARING PROBABILITY

In this section, we first review the results on optimal unambiguous discrimination problems with the knowledge of *a priori* preparing probabilities. According to what kind of classical knowledge can be utilized, the four cases are discussed in the three subsections. (1)Case A1, without classical knowledge of either state but with a single copy of unknown states; (2) Case A2, with only classical knowledge of one of the two states and a single copy of the other unknown state; (3)Case A3, with only classical knowledge of one of the two states and the absolute value of the inner product of both states, and also with a single copy of the other unknown state; (4) Case A4, with classical knowledge of both states.

The A1 and A4 cases will be investigated in subsection A and C, respectively, and the A2 and A3 cases will be studied in subsection B.

A. Optimal unambiguous discrimination problems without classical knowledge of discriminated states

In this subsection, we review the result of Ref. [7], and further discuss the optimal unambiguous discrimination problems in which the preparing probabilities is given, but none classical knowledge of discriminated states is available.

Given two unknown quantum states $|\psi_1\rangle$ and $|\psi_2\rangle$, we can construct a device to unambiguously discriminate between them. Two classically unknown states $|\psi_1\rangle$ and $|\psi_2\rangle$ are provided as two inputs for two program registers, respectively. Then we are given another qubit that is guaranteed to be one of two unknown states $|\psi_1\rangle$ and $|\psi_2\rangle$ stored in the two program registers. Our task is to determine, as best we can, which one the given qubit is. We are allowed to fail, but not to make a mistake. What is the best procedure to accomplish this? Our task is then reduced to the following measurement optimization problem.

One has two input states

$$|\Psi_1^{in}\rangle = |\psi_1\rangle_A|\psi_2\rangle_B|\psi_1\rangle_C; |\Psi_2^{in}\rangle = |\psi_1\rangle_A|\psi_2\rangle_B|\psi_2\rangle_C \quad (1)$$

where the subscripts A and B refer to the program registers, and the subscript C refers to the data register.

Our goal is to unambiguously distinguish between these inputs.

Let the elements of our POVM (positive-operator-valued measure) be Π_1, corresponding to unambiguously detecting $|\psi_1\rangle$, Π_2, corresponding to unambiguously detecting $|\psi_2\rangle$, and Π_0, corresponding to failure, respectively. The probabilities of successfully identifying the two possible input states are given by

$$\langle\Psi_1^{in}|\Pi_1|\Psi_1^{in}\rangle = p_1; \langle\Psi_2^{in}|\Pi_2|\Psi_2^{in}\rangle = p_2 \tag{2}$$

and the condition of no errors implies that

$$\Pi_1|\Psi_2^{in}\rangle = 0;\ \Pi_2|\Psi_1^{in}\rangle = 0 \tag{3}$$

In addition, because the alternatives represented by the POVM exhaust all possibilities, we have that

$$\Pi_1 + \Pi_2 + \Pi_0 = I \tag{4}$$

Since we have no classical knowledge about $|\psi_1\rangle$ and $|\psi_2\rangle$, the right way of constructing POVM operators is to take advantage of the symmetrical properties of the states. Denoting $|1\rangle$ and $|0\rangle$ as two vectors of a basis, we define the antisymmetric state

$$|\psi_{BC}^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_B|1\rangle_C - |1\rangle_B|0\rangle_C) \tag{5}$$

and

$$|\psi_{AC}^-\rangle = \frac{1}{\sqrt{2}}(|0\rangle_A|1\rangle_C - |1\rangle_A|0\rangle_C) \tag{6}$$

and introduce the projectors to the antisymmetric subspaces of the corresponding qubit as

$$\P_{BC}^{as} = |\psi_{BC}^-\rangle\langle\psi_{BC}^-|; \P_{AC}^{as} = |\psi_{AC}^-\rangle\langle\psi_{AC}^-| \tag{7}$$

We now can take for Π_1 and Π_2 operators

$$\Pi_1 = \lambda_1 I_A \otimes \P_{BC}^{as}; \Pi_2 = \lambda_2 I_B \otimes \P_{AC}^{as} \tag{8}$$

To assure that Π_1, Π_2 and $\Pi_0 = I - \Pi_1 - \Pi_2$ be semi-positive operators, the following constraints should be satisfied:

$$2 - \lambda_1 - \lambda_2 \geq 0;\ 1 - \lambda_1 - \lambda_2 + \frac{3}{4}\lambda_1\lambda_2 \geq 0 \tag{9}$$

After some calculations, we have $p_i = \frac{1}{2}\lambda_i(1-\beta^2)$, where $i = 1, 2$ and $\beta = |\langle\psi_1|\psi_2\rangle|$. Suppose that η_1 is the preparation probability of $|\psi_1\rangle$, the average success probability is $P = p_1\eta_1 + p_2(1-\eta_1)$.

Since we have knowledge of η_1, our task is reduced to designing $\lambda_1(\eta_1)$ and $\lambda_2(\eta_1)$ such that the following average success probability

$$P = \frac{1}{2}[\lambda_1\eta_1 + \lambda_2(1-\eta_1)](1-\beta^2) \tag{10}$$

is maximal with the constrains given by Eq. (9). This is to say, the loss function can be expressed as

$$J = \max\min_{\{\eta_1\}}\{\frac{1}{2}[\lambda_1\eta_1 + \lambda_2(1-\eta_1)](1-\beta^2)\} \tag{11}$$

In this case, the optimum success probability has been summarized as follows

$$P_0^{opt}(\beta,\eta_1) = \begin{cases} \frac{1}{2}(1-\eta_1)(1-\beta^2) & \eta_1 \leq \frac{1}{5} \\ \frac{2}{3}[1-\sqrt{\eta_1(1-\eta_1)}](1-\beta^2) & \frac{1}{5} \leq \eta_1 \leq \frac{4}{5} \\ \frac{1}{2}\eta_1(1-\beta^2) & \eta_1 \geq \frac{4}{5} \end{cases} \tag{12}$$

where the subscript 0 of P_0^{opt} means that we have no a priori classical knowledge of $|\psi_1\rangle$ and $|\psi_2\rangle$ and the corresponding optimal action parameters are given by

$$\lambda_1^{0,opt}(\eta_1) = \begin{cases} 0 & \eta_1 \leq \frac{1}{5} \\ \frac{2}{3}[2-\sqrt{\frac{1-\eta_1}{\eta_1}}] & \frac{1}{5} \leq \eta_1 \leq \frac{4}{5} \\ 1 & \eta_1 \geq \frac{4}{5} \end{cases} \tag{13}$$

$$\lambda_2^{0,opt}(\eta_1) = \begin{cases} 1 & \eta_1 \leq \frac{1}{5} \\ \frac{2}{3}[2-\sqrt{\frac{\eta_1}{1-\eta_1}}] & \frac{1}{5} \leq \eta_1 \leq \frac{4}{5} \\ 0 & \eta_1 \geq \frac{4}{5} \end{cases} \tag{14}$$

B. Optimal unambiguous discrimination problems with partial classical knowledge of discriminated states

In this subsection, we re-discuss the optimal unambiguous discrimination problem for the cases A2 and A3 from the view point of decision theory, and throw some new insights, which are different from those in Ref. [8, 9].

Given one known quantum state $|\psi_1\rangle$ and one unknown quantum state $|\psi_2\rangle$, we can construct a device that unambiguously discriminate between them. We shall consider the following problem which may be a simple version of a programmable state discriminator. The unknown state $|\psi_2\rangle$ is provided as an input for the program register. Then we are given another qubit that is guaranteed to be in the known state $|\psi_1\rangle$ or the unknown state $|\psi_2\rangle$ stored in the program register. Our task is to determine, as best we can, which one the given qubit is. As in case A1, we are allowed to fail, but not to make a mistake. What is the best procedure to accomplish this?

In line with Ref. [7], one can construct such a device by viewing this problem as a task in measurement optimization. The measurement is allowed to return an inconclusive result but never an erroneous one. Thus, it will be described by a POVM that will return outcome 1 (the unknown state stored in the data register matches the known state $|\psi_1\rangle$), 2 (the unknown state stored in the data register matches $|\psi_2\rangle$ in the program register), or 0 (we do not learn anything about the unknown state stored in the data register). Our task is then reduced to the following measurement optimization problem.

One has two input states

$$|\Psi_1^{in}\rangle = |\psi_2\rangle_A|\psi_1\rangle_B; |\Psi_2^{in}\rangle = |\psi_2\rangle_A|\psi_2\rangle_B \tag{15}$$

where the subscript A refers to the program register (A contains $|\psi_2\rangle$), and the subscript B refers to the data register. Our goal is to unambiguously distinguish between these inputs.

Let the elements of our POVM be Π_1, corresponding to unambiguously detecting $|\psi_1\rangle$, Π_2, corresponding to unambiguously detecting $|\psi_2\rangle$, and Π_0, corresponding to failure, respectively. The probabilities of successfully identifying the two possible input states are given by

$$\langle\Psi_1^{in}|\Pi_1|\Psi_1^{in}\rangle = p_1; \langle\Psi_2^{in}|\Pi_2|\Psi_2^{in}\rangle = p_2 \quad (16)$$

and the condition of no errors implies that

$$\Pi_1|\Psi_2^{in}\rangle = 0;\ \Pi_2|\Psi_1^{in}\rangle = 0 \quad (17)$$

In addition, because the alternatives represented by the POVM exhaust all possibilities, we have that

$$\Pi_1 + \Pi_2 + \Pi_0 = I \quad (18)$$

Since we know nothing about $|\psi_2\rangle$ but have the classical knowledge of $|\psi_1\rangle$, the right way of constructing POVM operators is to take advantage of the symmetrical properties of the state as well as the classical knowledge of $|\psi_1\rangle$. Denoting $|\psi_1^\perp\rangle$ as the unit vector orthogonal to $|\psi_1\rangle$, we define the antisymmetric state

$$|\psi_{AB}^-\rangle = \frac{1}{\sqrt{2}}(|\psi_1\rangle_A|\psi_1^\perp\rangle_B - |\psi_1^\perp\rangle_A|\psi_1\rangle_B) \quad (19)$$

and introduce the projectors to the antisymmetric subspaces of the corresponding qubit as

$$¶_{AB}^{as} = |\psi_{AB}^-\rangle\langle\psi_{AB}^-| \quad (20)$$

Furthermore, in terms of $|\psi_1\rangle$, $|\psi_1^\perp\rangle$ and the absolute value of inner product $\beta = |\langle\psi_1|\psi_2\rangle|$, we can obtain $|\psi_2\rangle = e^{i\delta}\beta|\psi_1\rangle + \sqrt{1-\beta^2}|\psi_1^\perp\rangle$ where δ and β are unknown real and $0 \le \beta \le 1$.

Even though β may be unknown, it can be clearly illustrated from the geometric point of view. Therefore, we still consider the success probabilities of unambiguously discriminating two states as a function of both the preparation probabilities and the absolute value of inner product β of two discriminated states.

By making full use of the knowledge of $|\psi_1\rangle$ and $|\psi_1^\perp\rangle$, we construct the measurement operators Π_1 and Π_2 to satisfy the no-error condition given by Eq.(17) as follows:

$$\Pi_1 = \lambda_1 ¶_{AB}^{as} \quad (21)$$

and

$$\Pi_2 = \lambda_2|\psi_1\rangle_A|\psi_1^\perp\rangle_{BB}\langle\psi_1^\perp|_A\langle\psi_1| + \lambda_3|\psi_1^\perp\rangle_A|\psi_1^\perp\rangle_{BB}\langle\psi_1^\perp|_A\langle\psi_1^\perp| \text{and} \quad (22)$$

where λ_1, λ_2 and λ_3 are undetermined nonnegative real numbers. Using the Eqs. (21) and (22), we have

$$p_1 = \langle\Psi_1^{in}|\Pi_1|\Psi_1^{in}\rangle = \frac{1}{2}\lambda_1(1-\beta^2) \quad (23)$$

$$p_2 = \langle\Psi_2^{in}|\Pi_2|\Psi_2^{in}\rangle = \lambda_2\beta^2(1-\beta^2) + \lambda_3(1-\beta^2)^2 \quad (24)$$

By assuming that the preparation probabilities of $|\psi_1\rangle$ and $|\psi_2\rangle$ are η_1 and η_2 (where $\eta_2 = 1-\eta_1$), respectively, we can define the average probability P of successfully discriminating two states as

$$P = [\frac{1}{2}\lambda_1\eta_1 + \lambda_2\beta^2\eta_2 + \lambda_3(1-\beta^2)\eta_2](1-\beta^2) \quad (25)$$

where $\beta = |\langle\psi_1|\psi_2\rangle|$, and our task is to maximize the performance Eq. (25) subject to the constraint that $\Pi_0 = I - \Pi_1 - \Pi_2$ is a positive operator. It can be demonstrated that one cannot maximize Eq. (25) everywhere simultaneously without the classical knowledge of β. Still, we can give some further analysis.

To assure that Π_0, Π_1 and Π_2 are positive operators, we have the following inequality constraints:

$$1 - \lambda_1 - \lambda_2 + \frac{1}{2}\lambda_1\lambda_2 \ge 0 \quad (26)$$

$$0 \le \lambda_i \le 1 (i = 1,2,3) \quad (27)$$

Subsequently, we will discuss our strategies for the A2 and A3 cases.

(i) For the A2 case, we have the knowledge of preparing probability η_1, but no knowledge of β.

Our strategy is to design $\lambda_1^{1,w\beta}(\eta)$, $\lambda_2^{1,w\beta}(\eta)$ and $\lambda_3^{1,w\beta}(\eta)$ to maximize the minimal performance

$$J = \max\min_{\{\beta\}}[\frac{1}{2}\lambda_1\eta_1 + \lambda_2(1-\eta_1) + (\lambda_3-\lambda_2)(1-\beta^2)(1-\eta_1)] \quad (28)$$

subject to the constraints described by Eqs. (26) and (27).

No matter what η_1 is, one should always choose $\lambda_3^{1,w\beta}(\eta_1) = 1$. As for $\lambda_1^{1,w\beta}(\eta_1)$ and $\lambda_2^{1,w\beta}(\eta_1)$, they have to depend on η_1.

In short, we have

$$\lambda_1^{1,w\beta}(\eta_1) = \begin{cases} 0 & \eta_1 \le \frac{1}{2} \\ 2(1-\sqrt{\frac{1-\eta_1}{\eta_1}}) & \frac{1}{2} \le \eta_1 \le \frac{4}{5} \\ 1 & \eta_1 \ge \frac{4}{5} \end{cases} \quad (29)$$

$$\lambda_2^{1,w\beta}(\eta_1) = \begin{cases} 1 & \eta_1 \le \frac{1}{2} \\ 2(1-\sqrt{\frac{\eta_1}{1-\eta_1}}) & \frac{1}{2} \le \eta_1 \le \frac{4}{5} \\ 0 & \eta_1 \ge \frac{4}{5} \end{cases} \quad (30)$$

$$\lambda_3^{1,w\beta}(\eta_1) = 1 \quad (31)$$

Furthermore, we obtain the actual optimum success probability in this strategy:

$$P_1^{w\beta}(\beta,\eta_1) = \begin{cases} P_{1_1}^{w\beta}(\beta,\eta_1) & \eta_1 \le \frac{1}{2} \\ P_{1_2}^{w\beta}(\beta,\eta_1) & \frac{1}{2} \le \eta_1 \le \frac{4}{5} \\ P_{1_3}^{w\beta}(\beta,\eta_1) & \eta_1 \ge \frac{4}{5} \end{cases} \quad (32)$$

with

$$P_{1_1}^{w\beta}(\beta,\eta_1)=(1-\eta_1)(1-\beta^2)) \quad (33)$$

$$P_{1_2}^{w\beta}(\beta,\eta_1)=[1+\beta^2(1-\eta_1)-(1+\beta^2)\sqrt{\eta_1(1-\eta_1)}](1-\beta^2 \quad (34)$$

$$P_{1_3}^{w\beta}(\beta,\eta_1)=(1-\frac{1}{2}\eta_1-\beta^2(1-\eta_1))(1-\beta^2) \quad (35)$$

where the subscript 1 of $P_1^{w\beta}$ means that we just have *a priori* classical knowledge of $|\psi_1\rangle$, one of two discriminated states, and the superscript $w\beta$ of $P_1^{w\beta}$ implies that the optimum success probability is defined in terms of the worst case for β.

(ii) With *a priori* classical knowledge of both $|\langle\psi_1|\psi_2\rangle|=\beta$ and η_1 in hand, our task in the third case is to get the optimum values $\lambda_1^{1+,opt}(\beta,\eta_1)$, $\lambda_2^{1+,opt}(\beta,\eta_1)$ and $\lambda_3^{1+,opt}(\beta,\eta_1)$ to optimize the average success probability

$$J=[\frac{1}{2}\lambda_1\eta_1+\lambda_2\beta^2(1-\eta_1)+\lambda_3(1-\beta^2)(1-\eta_1)](1-\beta^2) \quad (36)$$

subject to the constraints Eqs. (26) and (27).

After some calculations, we have

$$\lambda_1^{1+,opt}(\beta,\eta_1)=\begin{cases}0 & \eta_1\le\frac{\beta^2}{1+\beta^2}\\ 2(1-\beta\sqrt{\frac{1-\eta_1}{\eta_1}}) & \frac{\beta^2}{1+\beta^2}\le\eta_1\le\frac{4\beta^2}{1+4\beta^2}\\ 1 & \eta_1\ge\frac{4\beta^2}{1+4\beta^2}\end{cases} \quad (37)$$

$$\lambda_2^{1+,opt}(\beta,\eta_1)=\begin{cases}1 & \eta_1\le\frac{\beta^2}{1+\beta^2}\\ 2-\frac{1}{\beta}\sqrt{\frac{\eta_1}{1-\eta_1}} & \frac{\beta^2}{1+\beta^2}\le\eta_1\le\frac{4\beta^2}{1+4\beta^2}\\ 0 & \eta_1\ge\frac{4\beta^2}{1+4\beta^2}\end{cases} \quad (38)$$

and

$$\lambda_3^{1+,opt}(\beta,\eta_1)\equiv 1 \quad (39)$$

Taking Eq. (36) into consideration, we obtain the corresponding optimum success probabilities:

$$P_{1+}^{opt}(\beta,\eta_1)=\begin{cases}P_{1+_1}^{opt}(\beta,\eta_1) & \eta_1\le\frac{\beta^2}{1+\beta^2}\\ P_{1+_2}^{opt}(\beta,\eta_1) & \frac{\beta^2}{1+\beta^2}\le\eta_1\le\frac{4\beta^2}{1+4\beta^2}\\ P_{1+_3}^{opt}(\beta,\eta_1) & \eta_1\ge\frac{4\beta^2}{1+4\beta^2}\end{cases} \quad (40)$$

with

$$P_{1+_1}^{opt}(\beta,\eta_1)=(1-\eta_1)(1-\beta^2) \quad (41)$$

$$P_{1+_2}^{opt}(\beta,\eta_1)=[1+\beta^2(1-\eta_1)-2\beta\sqrt{\eta_1(1-\eta_1)}](1-\beta^2) \quad (42)$$

$$P_{1+_3}^{opt}(\beta,\eta_1)=[1-\frac{1}{2}\eta_1-\beta^2(1-\eta_1)](1-\beta^2) \quad (43)$$

where the subscript 1+ of P_{1+}^{opt} means that we have *a priori* classical knowledge of one of the two discriminated states and the absolute value of the inner product of the two states.

C. Optimal unambiguous discrimination problems with complete classical knowledge

In this subsection, we recall the result of Ref. [6, 10] for the A4 case.

If we have complete *a priori* classical knowledge of both $|\psi_1\rangle$ and $|\psi_2\rangle$, the measurement is performed on the detected qubit. One can select the detection operators as

$$\Pi_1=\lambda_1|\psi_2^\perp\rangle\langle\psi_2^\perp|;\Pi_2=\lambda_2|\psi_1^\perp\rangle\langle\psi_1^\perp| \quad (44)$$

Our task is to choose λ_1 and λ_2 based on *a priori* information such that the average success probability

$$P=[\lambda_1\eta_1+\lambda_2(1-\eta_1)](1-\beta^2) \quad (45)$$

is maximized.

To assure that Π_0, Π_1 and Π_2 are positive operators, we have the following inequality constraints:

$$1-\lambda_1-\lambda_2\beta^2\ge 0 \quad (46)$$

and

$$1-\lambda_1-\lambda_2+(1-\beta^2)\lambda_1\lambda_2\ge 0 \quad (47)$$

where $|\langle\psi_1|\psi_2\rangle|=\beta$.

Since we have knowledge of preparing probability η_1 and β, we will make the following decision

$$\lambda_1^{2,opt}(\beta,\eta_1)=\begin{cases}0 & \eta_1\le\frac{\beta^2}{1+\beta^2}\\ \frac{1}{1-\beta^2}(1-\beta\sqrt{\frac{1-\eta_1}{\eta_1}}) & \frac{\beta^2}{1+\beta^2}\le\eta_1\le\frac{1}{1+\beta^2}\\ 1 & \eta_1\ge\frac{1}{1+\beta^2}\end{cases} \quad (48)$$

$$\lambda_2^{2,opt}(\beta,\eta_1)=\begin{cases}1 & \eta_1\le\frac{\beta^2}{1+\beta^2}\\ \frac{1}{1-\beta^2}(1-\beta\sqrt{\frac{\eta_1}{1-\eta_1}}) & \frac{\beta^2}{1+\beta^2}\le\eta_1\le\frac{1}{1+\beta^2}\\ 0 & \eta_1\ge\frac{1}{1+\beta^2}\end{cases} \quad (49)$$

Furthermore, we can obtain the optimum success probability for this case (also as per Ref. [6, 10]).

$$P_2^{opt}(\beta,\eta_1)=\begin{cases}P_{2_1}^{opt}(\beta,\eta_1) & \eta_1\le\frac{\beta^2}{1+\beta^2}\\ P_{2_2}^{opt}(\beta,\eta_1) & \frac{\beta^2}{1+\beta^2}\le\eta_1\le\frac{1}{1+\beta^2}\\ P_{2_3}^{opt}(\beta,\eta_1) & \eta_1\ge\frac{1}{1+\beta^2}\end{cases} \quad (50)$$

with

$$P_{2_1}^{opt}(\beta,\eta_1)=(1-\eta_1)(1-\beta^2) \quad (51)$$

$$P_{2_2}^{opt}(\beta,\eta_1)=1-2\sqrt{\eta_1(1-\eta_1)}\beta \quad (52)$$

$$P_{2_3}^{opt}(\beta,\eta_1)=\eta_1(1-\beta^2) \quad (53)$$

where $|\langle\psi_1|\psi_2\rangle|=\beta$, the subscript 2 of P_2^{opt} means that we have the classical knowledge of both discriminated states.

III. OPTIMAL UNAMBIGUOUS DISCRIMINATION PROBLEMS WITHOUT *A PRIORI* PREPARING PROBABILITY

In this section, we will discuss various optimal unambiguous discrimination problems without *a priori* preparing probability. Corresponding to what have been explored in section III, we have also four cases taken into consideration as follows: (1) Case B1, without classical knowledge of either state but with a single copy of unknown states;(2) Case B2, with classical knowledge of one of the two states and a single copy of the other unknown state;(3) Case B3, with classical knowledge of one of the two states and the absolute value of the inner product of both states, and also with a single copy of the other unknown state; (4) Case B4, with classical knowledge of both states.

The B1 and B4 cases will be investigated in subsection A and C, respectively, and the B2 and B3 cases will be studied in subsection B.

A. Optimal unambiguous discrimination problems without classical knowledge of discriminated states

Since we have the same classical knowledge of discriminated states in this case as in Section II. A, we can follow the analysis in Section II. A and choose Π_1 and Π_2 as Eqs. (8).

To assure that Π_1, Π_2 and $\Pi_0 = I - \Pi_1 - \Pi_2$ be semi-positive operators, the constraints on λ_1 and λ_2 described by Eq.(9) should be satisfied.

However, since we have no knowledge of preparing probability, we have to design λ_1 and λ_2 without *a priori* information of η_1. Our strategy is to maximize the minimal performance

$$J = P_0^{w\eta_1}(\beta) = \max \min_{\{\eta_1\}} \frac{1}{2}[\lambda_1\eta_1 + \lambda_2(1-\eta_1)](1-\beta^2) \quad (54)$$

with the constraints in Eq. (9).

After careful calculations, we obtain that

$$\lambda_1^{0,w\eta_1} = \lambda_2^{0,w\eta_1} = \frac{2}{3} \quad (55)$$

Substituting Eq. (55) into Eq. (54) yields

$$P_0^{w\eta_1}(\beta) = \frac{1}{3}(1-\beta^2) \quad (56)$$

B. Optimal unambiguous discrimination problems with partial classical knowledge of discriminated states

In this subsection, we will discuss the optimal unambiguous discrimination problems for the B2 and B3 cases where partial classical knowledge but none knowledge of preparing probabilities of discriminated states are available.

Since we have the same partial classical knowledge of discriminated states in this section as in Section II.B, we can follow the analysis in Section II.B and choose Π_1 and Π_2 as Eqs.(21) and (22).

To assure that Π_1, Π_2 and $\Pi_0 = I - \Pi_1 - \Pi_2$ be semi-positive operators, the constraints on λ_1 and λ_2 described by (26) and (27) should be satisfied.

Our task is to design λ_1 and λ_2 such that the average success probability

$$P = [\frac{1}{2}\lambda_1\eta_1 + \lambda_2\beta^2\eta_2 + \lambda_3(1-\beta^2)\eta_2](1-\beta^2) \quad (57)$$

is maximized.

Subsequently, we will discuss our strategies for the B2 and B3 cases, respectively.

(i) If we have neither the knowledge of preparing probabilities nor the knowledge of β, our task is reduced to designing $\lambda_1^{1,w\beta\eta_1}$, $\lambda_2^{1,w\beta\eta_1}$ and $\lambda_3^{1,w\beta\eta_1}$ to maximize the minimal performance

$$J = \max \min_{\{\beta,\eta_1\}} [\frac{1}{2}\lambda_1\eta_1 + \lambda_2(1-\eta_1) + (\lambda_3-\lambda_2)(1-\beta^2)(1-\eta_1)] \quad (58)$$

subject to the constraints in Eqs. (26) and (27).

Following some similar calculations in the subsection II. B, we have the optimal actions as follows

$$\lambda_1^{1,w\beta\eta_1} = 3-\sqrt{5}; \lambda_2^{1,w\beta\eta_1} = \frac{1}{2}(3-\sqrt{5}); \lambda_3^{1,w\beta\eta_1} = 1 \quad (59)$$

By substituting them into Eq. (25), we get the actual success probability with regard to this strategy:

$$P_1^A(\beta,\eta_1) = [\frac{3-\sqrt{5}}{2} + \frac{\sqrt{5}-1}{2}(1-\beta^2)(1-\eta_1)](1-\beta^2) \quad (60)$$

and the optimum success probability in the worst case:

$$P_1^{w\beta\eta_1}(\beta) = \frac{3-\sqrt{5}}{2}(1-\beta^2) \quad (61)$$

where the subscript 1 of $P_1^{w\beta\eta_1}$ means that we just have *a priori* classical knowledge of $|\psi_1\rangle$, one of two discriminated states, and the superscript $w\beta\eta_1$ implies that the optimum success probability is defined in terms of the worst case for both β and η_1.

(ii) For the B3 case, we have the knowledge of β, but no knowledge of preparing probability η_1.

Our task is to design $\lambda_1^{1+,w\eta_1}(\beta)$, $\lambda_2^{1+,w\eta_1}(\beta)$ and $\lambda_3^{1+,w\eta_1}(\beta)$ to maximize the minimal performance

$$J = \max \min_{\{\eta_1\}} [\frac{1}{2}\lambda_1\eta_1 + \lambda_2(1-\eta_1) + (\lambda_3-\lambda_2)(1-\beta^2)(1-\eta_1)] \quad (62)$$

subject to the constraints given by Eqs. (26) and (27).

After some calculation, we have

$$\lambda_1^{1+,w\eta_1}(\beta) = \begin{cases} 1 & \beta \leq \frac{\sqrt{2}}{2} \\ \beta^2 + 2 - \sqrt{\beta^4 + 4\beta^2} & \beta \geq \frac{\sqrt{2}}{2} \end{cases} \quad (63)$$

$$\lambda_2^{1+,w\eta_1}(\beta) = \begin{cases} 0 & \beta \leq \frac{\sqrt{2}}{2} \\ \frac{3}{2} - \sqrt{\frac{1}{4} + \frac{1}{\beta^2}} & \beta \geq \frac{\sqrt{2}}{2} \end{cases} \quad (64)$$

$$\lambda_3^{1+,w\eta_1}(\beta) \equiv 1 \quad (65)$$

Furthermore, we obtain the actual success probability:

$$P_{1+}^{A}(\beta, \eta_1) = \begin{cases} \{\frac{1}{2} + (\frac{1}{2} - \beta^2)(1-\eta_1)\}(1-\beta^2) & \beta \leq \frac{\sqrt{2}}{2} \\ (1 + \frac{1}{2}\beta^2 - \frac{1}{2}\sqrt{\beta^4 + 4\beta^2})(1-\beta^2) & \beta \geq \frac{\sqrt{2}}{2} \end{cases} \quad (66)$$

and optimum success probabilities in the worst case

$$P_{1+}^{w\eta_1}(\beta) = \begin{cases} \frac{1}{2}(1-\beta^2) & \beta \leq \frac{\sqrt{2}}{2} \\ (1 + \frac{1}{2}\beta^2 - \frac{1}{2}\sqrt{\beta^4 + 4\beta^2})(1-\beta^2) & \beta \geq \frac{\sqrt{2}}{2} \end{cases} \quad (67)$$

C. Optimal unambiguous discrimination problems with complete classical knowledge of discriminated states

This subsection discuss the optimal unambiguous discrimination problem where complete classical knowledge of discriminated states but none *a priori* probabilities of preparing the discriminated states are available.

Here we have the same classical knowledge of discriminated states in this case as in Section II. C, thus we can follow the analysis in Section II. C and choose Π_1 and Π_2 as Eq. (44).

In order to assure that Π_1, Π_2 and $\Pi_0 = I - \Pi_1 - \Pi_2$ be semi-positive, the constraints on λ_1 and λ_2 given by Eqs. (46) and (47) should be satisfied where $|\langle\psi_1|\psi_2\rangle| = \beta$.

And what we shall do here is the same, i.e., to choose λ_1 and λ_2 based on *a priori* information such that the average success probability

$$P = [\lambda_1\eta_1 + \lambda_2(1-\eta_1)](1-\beta^2) \quad (68)$$

with the constraints in Eqs. (46) and (47).

When we have no knowledge of preparing probability η_1, our task is to choose $\lambda_1^{2,w\eta_1}(\beta)$ and $\lambda_2^{2,w\eta_1}(\beta)$ to optimize the following performance

$$J = \max \min_{\{\eta_1\}} [\lambda_1\eta_1 + \lambda_2(1-\eta_1)](1-\beta^2) \quad (69)$$

with the constraints described by Eqs. (46) and (47).

In this case, we have

$$\lambda_1^{2,w\eta_1}(\beta) = \lambda_2^{2,w\eta_1}(\beta) = \frac{1}{1+\beta} \quad (70)$$

$$P_2^{w\eta_1}(\beta) = 1 - \beta \quad (71)$$

where the subscript 2 of $P_2^{w\eta_1}$ means that we have the classical knowledge of both discriminated states, and the superscript $w\eta_1$ implies that the optimum success probability is defined in terms of the worst case for η_1.

IV. CONCLUSION

From the aforementioned results, we demonstrate that there are two types of *a priori* knowledge in optimum ambiguous state discrimination problems: *a priori* knowledge of discriminated states themselves and *a priori* probabilities of preparing the states. It is demonstrated that both types of *a priori* knowledge can be utilized to improve the optimum average success probabilities. It is very interesting to find that both types of discriminators and the constraint conditions of action spaces are decided just by the classical knowledge of discriminated states. This is in contrast to the observation that both the loss functions (optimum average success probabilities) and optimal decisions depend on two types of *a priori* knowledge. The detailed discussion for the role of a priori knowledge in the optimization of quantum information processing can be found in our recent paper[11]

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