PHASE ENTRAINMENT IN COUPLED TIME-DELAY SYSTEMS

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Abstract

The notion of phase and phase synchronization in time-delay systems are not well understood despite of our substantial understanding of these phenomena in non-delay (low-dimensional) systems. We will discuss briefly the drawbacks in estimating phase in time-delay systems usually exhibiting highly non-phase-coherent hyperchaotic attractors with complex topological properties. We will provide a brief note on the different approaches that we have used in estimating phase and subsequently identifying chaotic phase synchronization in coupled time-delay systems. Further, we will demonstrate the existence of chaotic phase synchronization in paradigmatic models such as a piecewise linear time-delay systems and Mackey-Glass systems numerically and experimentally in time-delay electronic circuits with a threshold nonlinearity.

Key words

Phase synchronization, time-delay systems, non-phase-coherent attractor, Nonlinear transformation, Recurrence quantification measures, Localized sets.

1 Introduction

Synchronization of chaotic systems driven by common signals have been an area of extensive research since the pioneering works of Fujisaka and Yamada [Fujisaka and Yamada (1983)] and of Pecora and Carroll [Pecora and Carroll, 1990]. Since the identification of complete (identical) chaotic synchronization, different kinds of chaotic synchronizations have been identified and demonstrated both theoretically and experimentally (cf. [Pikovsky, Rosenblum and Kurths, 2001; Boccaletti, Kurths, Osipov, Valladares and Zhou, (2002); Lakshmanan and Senthilkumar, (2010)]). Recently, synchronization in coupled time-delay systems with or without time-delay coupling has become an active area of research by exploiting the infinite dimensional nature of the underlying systems, which exhibits a large number of positive Lyapunov exponents as a function of the delay time, for potential applications [Shahverdiev, Sivaprakasam and Shore, (2002); Boccaletti, Pecora and Pelaez, (2001); Cuomo and Oppenheim (1993); Hayes, Grebogy, Ott and Mark, (1994); Garcia-Ojalvo and Roy, (2001); VanWiggeren and Roy, (2002); Senthilkumar, Lakshmanan and Kurths, (2006); Senthilkumar, Lakshmanan and Kurths, (2007); Senthilkumar, Lakshmanan and Kurths, (2008); Senthilkumar, Kurths and Lakshmanan, (2009); Suresh, Senthilkumar, Lakshmanan and Kurths, (2010); Senthilkumar, Srinivasan, Murali, Lakshmanan and Kurths, (2010); Srinivasan, Senthilkumar, Murali, Lakshmanan and Kurths, (2011)].

Among the different kinds of synchronization, chaotic phase synchronization (CPS) refers to the coincidence of characteristic time scales of interacting chaotic dynamical systems, while their amplitudes remain chaotic and often uncorrelated [Pikovsky, Rosenblum and Kurths, 2001; Boccaletti, Kurths, Osipov, Valladares and Zhou, (2002)]. CPS plays a crucial role in understanding a large class of weakly interacting nonlinear dynamical systems and has been demonstrated both theoretically and experimentally in a wide variety of natural systems [Pikovsky, Rosenblum and Kurths, 2001; Boccaletti, Kurths, Osipov, Valladares and Zhou, (2002)]. Despite our substantial understanding of the phenomenon of CPS and its potential applications in low-dimensional systems, only a very few studies on it have been reported in time-delayed systems, which are essentially infinite-dimensional in nature [Senthilkumar, Lakshmanan and Kurths, (2006); Senthilkumar, Lakshmanan and Kurths, (2007); Senthilkumar, Lakshmanan and Kurths, (2008); Senthilkumar, Kurths and Lakshmanan, (2009); Suresh, Senthilkumar, Lakshmanan and Kurths, (2010); Srinivasan, Senthilkumar, Murali, Lakshmanan and Kurths, (2011)]. Due to the highly non-phase-coherent chaotic/hyperchaotic attractors with complex topological properties exhibited by these systems in general, it is often impossible to estimate the phase explicitly and to identify CPS in time-delay systems.

Nevertheless, we have introduced a nonlinear transformation to recast the original non-phase-coherent attractors into smeared limit-cycle attractors to enable to estimate the phase explicitly and to identify CPS in time-delay model systems for the first time in the literature [Senthilkumar, Lakshmanan and Kurths, (2006)]. This study has initiated further investigations in identifying and understanding the mechanism of phase synchronization transition in coupled time-delay systems [Senthilkumar, Lakshmanan and Kurths, (2006); Senthilkumar, Lakshmanan and Kurths, (2007); Senthilkumar, Lakshmanan and Kurths, (2008); Senthilkumar, Kurths and Lakshmanan, (2009); Suresh, Senthilkumar, Lakshmanan and Kurths, (2010); Srinivasan, Senthilkumar, Murali, Lakshmanan and Kurths, (2011)]. In particular, phase synchronization in an array of time-delay systems along with their mechanism of transition to global phase synchronization has been reported [Suresh, Senthilkumar, Lakshmanan and Kurths, (2010)], first experimental demonstration of phase synchronization in coupled time-delay systems with threshold nonlinearity has also been reported [Senthilkumar, Srinivasan, Murali, Lakshmanan and Kurths, (2010)].

In this paper, we provide a brief review of our important results on phase synchronization in coupled time-delay systems including an experimental confirmation using electronic circuits. The plan of the paper is as follows. In Sec. 2, we provide a brief note on the estimates of phase that we have applied for time-delay systems in estimating phase and subsequently identifying chaotic phase synchronization. We demonstrate the existence of CPS in paradigmatic models such as a piecewise linear time-delay systems and Mackey-Glass systems numerically and experimentally in a piecewise linear time-delay systems with a threshold nonlinearity in Sec. 3 and endup with summary and conclusion in Sec. 4.

2 Estimates of phase applied for time-delay systems

We usually encounter with the terminologies phase-coherent and non-phase-coherent chaotic attractors while studying CPS. If the flow of a dynamical system has a proper rotation around a fixed reference point, then the corresponding attractor is termed as phase-coherent attractor. In contrast, if the flow does not have a proper rotation around a fixed reference point then the corresponding attractor is called as non-phase-coherent attractor. While methods have been well established in the literature to identify phase and to study CPS in phase-coherent chaotic attractors, methods to identify phase of non-phase-coherent chaotic attractors have not yet been well established. Even the most promising approach available in the literature to calculate the phase of non-phase-coherent attractors is based on the concept of curvature [Osipov, Hu, Zhou, Ivanchenko and Kurths, (2003)], but this is often restricted to low-dimensional systems. However, we find that this procedure does not work in the case of nonlinear time-delay systems in general, where very often the attractor is non-phase-coherent and high-dimensional. Unfortunately methods to identify phase and to study CPS in time-delay systems, which often exhibit highly complicated hyperchaotic attractors, have not yet been well understood. Hence defining and estimating phase from the hyperchaotic attractors of the time-delay systems itself is a challenging task and so specialized techniques/tools have to be identified to introduce the notion of phase in such systems. In the following, we account briefly on the different measures that we have employed in identifying CPS in coupled time-delay systems.

2.1 Transformation of the original attractor

We have introduced a nonlinear transformation to rescale the original non-phase-coherent chaotic attractor into smeared limit cycle like attractor with a single center of rotation. The transformation is performed by introducing a new state variable [Senthilkumar, Lakshmanan and Kurths, (2006); Senthilkumar, Kurths and Lakshmanan, (2009)],

$$z(t+\tau) = x(t)x(t+\hat{\tau})/x(t+\tau), \qquad (1)$$

where $\hat{\tau}$ is the optimal value of time delay to be chosen in order to avoid any additional center of rotation. This functional form of the transformation (along with a delay time $\hat{\tau}$) has been identified by generalizing the transformation used in the case of chaotic attractors in the Lorenz systems [Pikovsky, Rosenblum and Kurths, 2001]. Now, the projected trajectory in the new state space $(x(t+\tau), z(t+\tau))$ will results in a smeared limit cycle like attractor with a single fixed center of rotation. Conventional approaches for estimating phase of phase-coherent attractors can now be applied to the transformed attractor in the new state space to estimate the phase explicitly and to identify CPS in time-delay systems with non-phase-coherent attractors. The other estimates that we have employed to identify CPS does not involve estimation of the phase explicitly but instead provides a qualitative and quantitative confirmation of existence of CPS.

It is to be noted that the above transformation (1) can be applied to the non-phase-coherent attractors of any time-delay system in general, except for the fact that the optimal value of $\hat{\tau}$ should be chosen for each system appropriately through trial and error by requiring the geometrical structure of the transformed attractor to have a fixed center of rotation. We have adopted here a geometric approach for the selection of $\hat{\tau}$ and look for an optimum transform which leads to a phase-coherent structure. This is indeed demonstrated for the attractor of piece-wise linear and Mackey-Glass time-delay systems in the next section [Senthilkumar, Lakshmanan and Kurths, (2006); Senthilkumar, Kurths and Lakshmanan, (2009)]. The main point that we want to emphasize here is that even for highly non-phase-coherent hyperchaotic attractors of time-delay systems, there is every possibility to identify suitable transformations of the type (1) to unfold the attractor and to identify phase as will demonstrate for two paradigmatic time-delay systems. However, formulating a more generalized transformation to include non-phase-coherent attractors of a large class of time-delay systems remains an open problem.

2.2 Recurrence measures

The complex synchronization phenomena in the coupled time-delay systems can also be analyzed by means of methods based on recurrence plots [Romano, Thiel, Kurths, Kiss and Hudson, (2005); Marwan, Romano, Thiel and Kurths, J (2007)]. These methods help to identify and quantify CPS particularly in non-phase coherent attractors. For this purpose, the generalized autocorrelation function $P(t)$ has been introduced in [Romano, Thiel, Kurths, Kiss and Hudson, (2005); Marwan, Romano, Thiel and Kurths, J (2007)] as

$$P(t) = \frac{1}{N-t}\sum_{i=1}^{N-t}\Theta(\epsilon - ||X_i - X_{i+t}||), \quad (2)$$

where Θ is the Heaviside function, X_i is the ith data corresponding to either the drive variable or the response variable and ϵ is a predefined threshold. $||.||$ is the Euclidean norm and N is the number of data points. $P(t)$ can be considered as a statistical measure about how often ϕ has increased by 2π or multiples of 2π within the time t in the original space. If two systems are in CPS, their phases increase on average by $K.2\pi$, where K is a natural number, within the same time interval t. The value of K corresponds to the number of cycles when $||X(t+T) - X(t)|| \sim 0$, or equivalently when $||X(t+T) - X(t)|| < \epsilon$, where T is the period of the system. Hence, looking at the coincidence of the positions of the maxima of $P(t)$ for both systems, one can qualitatively identify CPS.

A criterion to quantify CPS is the cross correlation coefficient between the drive, $P_1(t)$, and the response, $P_2(t)$, which can be defined as Correlation of Probability of Recurrence (CPR)

$$CPR = \langle \bar{P}_1(t)\bar{P}_2(t)\rangle / \sigma_1\sigma_2, \quad (3)$$

where $\bar{P}_{1,2}$ means that the mean value has been subtracted and $\sigma_{1,2}$ are the standard deviations of $P_1(t)$ and $P_2(t)$ respectively. If both systems are in CPS, the probability of recurrence is maximal at the same time t and CPR ≈ 1. If they are not in CPS, the maxima do not occur simultaneously and hence one can expect a drift in both the probability of recurrences and low values of CPR.

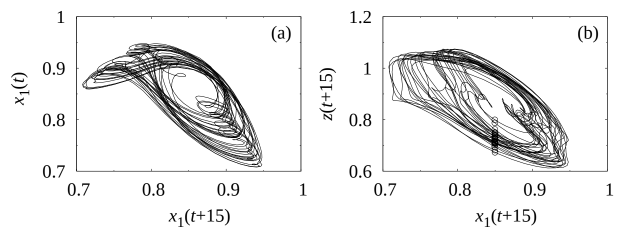

Figure 1. (a) The non-phase coherent hyperchaotic attractor of the uncoupled piecewise linear time-delay system for the parameter values $a = 1.0, b = 1.2$ and $\tau = 15$. and (b) Transformed attractor in the $x_1(t+\tau)$ and $z(t+\tau)$ space. Here the Poincaré points are represented as open circles.

2.3 Concept of Localized sets

Another interesting framework to identify CPS is the concept of localized sets [Pereira, Baptista and Kurths, (2007)]. This approach provides an easy and efficient way to detect CPS especially in complicated non-phase-coherent attractors. The basic idea of this concept is to define a typical event in one of the systems and then observe the other system whenever this event occurs. These observations give raise to a set D. Depending upon the property of this set D, one can state whether PS exists or not. The coupled systems evolve independently if the sets obtained by observing the corresponding events in the systems spread over the attractor of the systems. On the other hand, if the sets are localized on the attractors then CPS exists between them.

2.4 Lyapunov exponents

Transition of zero Lyapunov exponent of the response system to negative values is used to characterize the onset of phase synchronization in the low-dimensional systems. Chaotic phase synchronization in time-delay systems can also be characterized by the transitions in the Lyapunov exponents of the coupled time-delay systems [Senthilkumar, Lakshmanan and Kurths, (2006); Senthilkumar, Kurths and Lakshmanan, (2009)].

3 CPS in coupled time-delay systems

We consider the following unidirectionally coupled drive $x_1(t)$ and response $x_2(t)$ systems,

$$\dot{x}_1(t) = -ax_1(t) + b_1 f(x_1(t-\tau)), \quad (4a)$$
$$\dot{x}_2(t) = -ax_2(t) + b_2 f(x_2(t-\tau)) + b_3 f(x_1(t-\tau)), \quad (4b)$$

where b_1, b_2 and b_3 are constants, $a > 0$, τ is the delay time and $f(x)$ is an appropriate nonlinear function. Now we will demonstrate the existence of CPS in three different protype time-delay systems.

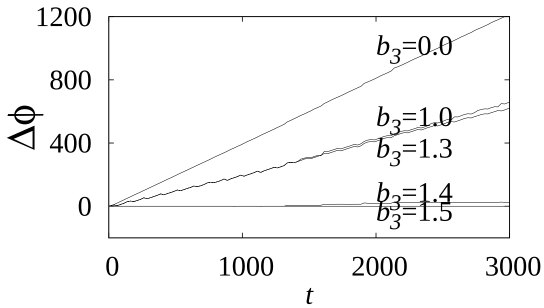

Figure 2. Phase differences ($\Delta\phi = \phi_1^z(t) - \phi_2^z(t)$) between the coupled piecewise linear time-delay systems for different values of the coupling strength $b_3 = 0.0, 1.0, 1.3, 1.4$ and 1.5.

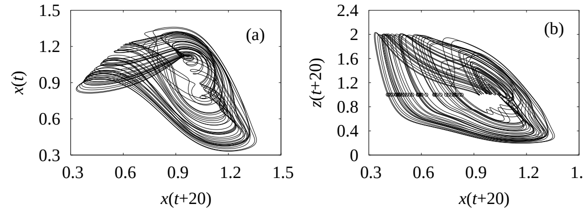

Figure 3. (a) The non-phase coherent chaotic attractor of the Mackey-Glass time-delay system for the parameter values $a = 0.1, b_1 = 0.2$ and $\tau = 20$ and (b) Transformed attractor in the $x_1(t+\tau)$ and $z(t+\tau)$ space along with the Poincaré points represented as open circles.

3.1 CPS in coupled piecewise linear time-delay systems

The nonlinear function $f(x)$ in the above coupled time-delay systems corresponding to the piecewise linear time-delay systems is an odd piecewise linear function defined as

$$f(x) = \begin{cases} 0, & x \leq -4/3 \\ -1.5x - 2, & -4/3 < x \leq -0.8 \\ x, & -0.8 < x \leq 0.8 \\ -1.5x + 2, & 0.8 < x \leq 4/3 \\ 0, & x > 4/3 \end{cases} \tag{5}$$

We have chosen the value of the parameters as $a = 1.0, b_1 = 1.2, b_2 = 1.1$ and $\tau = 15$. For this parametric choice, in the absence of coupling, the drive $x_1(t)$ and the response $x_2(t)$ systems evolve independently, which exhibit hyperchaotic attractors (Fig. 1) for the chosen parameter values [Senthilkumar, Lakshmanan and Kurths, (2006)]. The original and the transformed hyperchaotic attractors of the piecewise linear time-delay system are shown in Fig. 1(a) and (b), respectively. Open circles in Fig. 1(b) correspond to the Poincaré points of the smeared limit-cycle-like attractor. We find the optimal value of $\hat{\tau}$ for the transformation specified by Eq. (1) for the attractor (Fig. 1(a)) of the piecewise linear time-delay system to be 1.6. It is to be noted that on closer examination of the transformed attractor (Fig. 1(b)) in the vicinity of the common center, it does not have any closed loop, unlike the case of the original attractor, even though the trajectories show sharp turns in some regime of the phase space.

Now, the phase of the transformed attractor can be defined based on the Poincaré section technique [Pikovsky, Rosenblum and Kurths, 2001]. Phases, $\phi_1^z(t)$ and $\phi_2^z(t)$, of the drive $x_1(t)$ and the response $x_2(t)$ systems, respectively, are calculated from the state variables $z_1(t+\tau)$ and $z_2(t+\tau)$. The phase differences ($\Delta\phi = \phi_1^z(t) - \phi_2^z(t)$) between the drive and the response systems are shown in Fig. 2 for different values of the coupling strength b_3. The phase difference $\Delta\phi$ between the coupled piecewise linear time-delay systems for $b_3 = 0.0$ (uncoupled) increases monotonically as a function of time confirming that both systems are in an asynchronous state (also nonidentical) in the absence of coupling between them. For the values of $b_3 = 1.0$ and 1.3, the phase slips in the corresponding phase difference $\Delta\phi$ show that the systems are in a transition state. The strong boundedness of the phase difference is obtained for $b_3 > 1.382$ and it becomes zero for the value of the coupling strength $b_3 = 1.5$, showing a high quality CPS [Senthilkumar, Lakshmanan and Kurths, (2006)].

3.2 CPS in coupled Mackey-Glass time-delay systems

The nonlinear function $f(x)$ in the coupled time-delay systems, Eq. (4), corresponding to the Mackey-Glass time-delay systems is represented as

$$f(x) = x(t-\tau)/(1.0 + x(t-\tau)^{10}). \tag{6}$$

We have chosen the parameter values as $a = 0.1, b_1 = 0.2, b_2 = 0.205$ and $\tau = 20$. The non-phase-coherent chaotic attractor of the uncoupled Mackey-Glass system for the above choice of parameters is shown in Fig. 3(a). The transformed attractor, for the optimal value of the delay time $\hat{\tau} = 8.0$ in Eq. (1), is depicted in Fig. 3(b). The Poincaré points are shown as open circles in the Fig. 3(b) from which the instantaneous phase $\phi_1^z(t)$ is calculated using the Poincaré section technique. The phase differences $\Delta\phi = \phi_1^z(t) - \phi_2^z(t)$ between the coupled Mackey-Glass systems for the values of the coupling strength $b_3 = 0.04, 0.08, 0.11, 0.12$ and 0.3 are shown in Fig. 4. For the value of the coupling strength $b_3 = 0.3$, there exists a strong boundedness in the phase difference indicating the existence of CPS [Senthilkumar, Lakshmanan and Kurths, (2007)].

The existence of CPS is also confirmed from the original non-transformed attractors of the coupled systems using the recurrence quantification measures, Lyapunov exponents and the concept of Localized sets [Lakshmanan and Senthilkumar, (2010); Senthilkumar, Lakshmanan and Kurths, (2006); Senthilkumar, Lakshmanan and Kurths, (2007)]. Further, the existence of CPS in coupled Ikeda systems exhibiting more complex hyperchaotic attractors is also

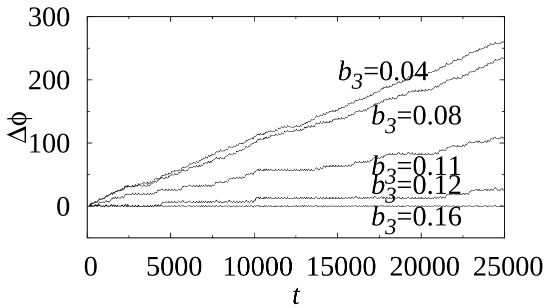

Figure 4. Phase differences ($\Delta\phi = \phi_1^z(t) - \phi_2^z(t)$) between the coupled Mackey-Glass systems for different values of the coupling strength $b_3 = 0.04, 0.08, 0.11, 0.12$ and 0.3.

demonstrated [Senthilkumar, Lakshmanan and Kurths, (2008)].

3.3 Experimental confirmation of CPS

In this section, we will discuss briefly the experimental results on CPS in coupled time-delay systems with threshold nonlinearity. The details of circuit and its parameter values can be seen in Ref. [Srinivasan, Senthilkumar, Murali, Lakshmanan and Kurths, (2011); Srinivasan, Senthilkumar, Murali, Lakshmanan and Kurths, (2011)]. The normalized coupled equation is identical to the Eqs. (4) with the following form of threshold nonlinearity

$$f(x) = Af^* - Bx. \tag{7a}$$

Here

$$f^* = \begin{cases} -x^* & x < -x^*, \\ x & -x^* \le x \le x^*, \\ x^* & x > x^*, \end{cases} \tag{7b}$$

where x^* is a controllable threshold value. The estimated normalized values turn out to be $x^* = 0.7$, $A = 5.2$, $B = 3.5$, $b_1 = 1.2$ and $b_2 = 1.1$ in accordance with the values of the circuit elements. In the following, we will demonstrate the existence of CPS as a function of the coupling strength ε in both chaotic and hyperchaotic regimes for suitable values of the delay time τ. The snapshots of the time series of both drive and response systems as seen from the oscilloscope are shown in Fig. 5(a) in the chaotic regime for the delay time $\tau = 1.33$ and the coupling strength $\varepsilon = 0.9$, indicating the evolution of both systems in-phase with each other. Similarly, the snapshots of the time series evolving in-phase with each other in the hyperchaotic regime for the delay time $\tau = 6.0$ are shown in Fig. 5(b) for $\varepsilon = 0.7$.

The existence of CPS is further characterized experimentally by using the framework of localized sets [Pereira, Baptista and Kurths, (2007)]. The sets

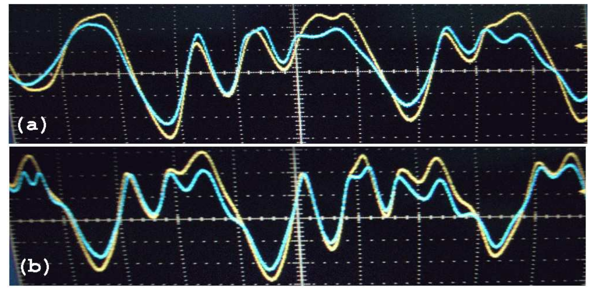

Figure 5. (color online) Snapshots of the time evolution of both coupled systems indicating the existence of CPS in (a) chaotic regime and (b) hyperchaotic regime.

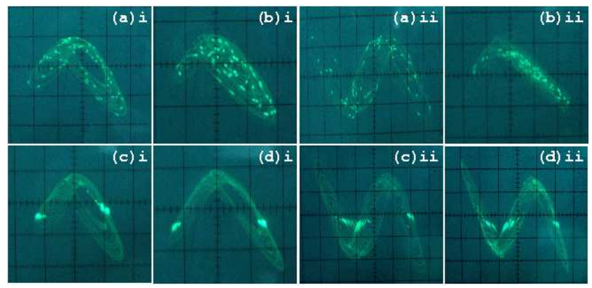

Figure 6. (color online) Experimental characterization of CPS using the framework of localized sets in the chaotic regime (i) for $\tau = 1.33$ and in the hyperchaotic regime (ii) for $\tau = 6.0$. Sets in the drive and the response systems are distributed in (a) and (b) for $\varepsilon = 0.3$ indicating the asynchronous state and localized in (c) and (d) for $\varepsilon = 0.9$ indicating CPS, respectively.

obtained by sampling the time series of one of the systems whenever a maximum occurs in the other one are plotted along with the chaotic attractor of the same system for the delay time $\tau = 1.33$ in Fig. 6. The sets distributed over the entire attractor of both the drive (Fig. 6(a)i) and the response (Fig. 6(b)i) systems for the coupling strength $\varepsilon = 0.3$ indicate that the time-delay systems evolve independently. The sets that are localized on the chaotic attractor of both the drive (Fig. 6(c)i) and the response (Fig. 6(d)i) systems for the coupling strength $\varepsilon = 0.9$ correspond to a perfect locking of the phases of both systems.

For rather small ε, the sets spread over the entire hyperchaotic attractors, for $\tau = 6.0$, of the drive and the response systems as shown in Figs. 6(a)ii and 6(b)ii, respectively, for $\varepsilon = 0.3$, which confirm that both systems evolve independently. On the other hand, for $\varepsilon = 0.9$, the observed sets that are localized on the attractors of the drive and the response systems as shown in Figs. 6(c)ii and 6(d)ii respectively, indeed confirm the existence of CPS in the hyperchaotic regime.

4 Conclusion

We have identified and characterized the existence of CPS in prototype time-delay systems such as a piecewise-linear time-delay system, the Mackey-Glass and experimentally in a piecewise-linear time-delay with threshold nonlinearity possessing highly non-phase-coherent chaotic attractors. We have shown that there is a typical transition from a non-synchronized state to CPS from the phase differences estimated from the transformed variables of the coupled systems using the nonlinear transformation. The existence of CPS is observed experimentally from snapshots of the time evolution of both the coupled systems and is confirmed with the framework of localized sets.

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